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2-LINEAR OPERATORS ON 2-MODULAR SPACES

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Abstract

In this paper, we observe some topological properties of 2-modular spaces. Further, we introduce and characterize a 2-ρ-bounded 2-linear operator from a 2-modular space into a normed space as well.

1. Introduction and Preliminaries

A modular space has important roles and applications in many areas, such as engineering, physics, economics, social sciences, etc. Therefore, it gains a lot of attention of many researchers from many fields. A concept of

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modular spaces was firstly initiated by Nakano in 1950 (see [6, 8, 12]). Later on, Mazur and Orlicz [7] and Musielak and Orlicz [9] modified the definition of the modular space proposed by Nakano, by avoiding the lattice structure in the space X on which the modular is defined as well as the monotonicity axiom for the modular.

As usual, the symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^*$ denote the natural number system, the real number system and the extended real number system, respectively. As given in [9], we can rewrite the definition of the modular as the following. Let X be a real linear space over $\mathbb{R}$. A nonnegative function $\rho : X \to \mathbb{R}^*$ is called a *modular* if for every $x, y \in X$, the following conditions hold:

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(-x) = \rho(x)$, and

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

If the condition (iii) is replaced by

(iii′) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$,

then the modular ρ is called a *convex modular*. A real linear space X equipped with a modular ρ, written (X, ρ) or X in short, is called a *modular space*.

Based on the definition of a modular as given above, we can easily check that every norm is a modular. Therefore, we can consider a modular as a generalization of a norm. As consequences, many concepts in normed spaces can be generalized into modular spaces.

In an earlier paper ([2] and [3]), Gahler introduced a concept of 2-norm spaces and *n*-norm spaces. One knows that every *n*-norm can define an $n-1$-norm. See [4] and [5]. Inductively, from an *n*-norm, we can derive a norm. Further, based on the theory of Gahler, Chu et al. [1] characterized 2-isometries on 2-norm spaces. Srivastava et al. [11] characterized linear *n*-functionals in *n*-norm spaces. Moreover, they formulated the extension of Hanh-Banach theorem for linear *n*-functionals in *n*-norm spaces.

Modular spaces are closed related to normed spaces [12]. Meanwhile, as mentioned before, any *n*-norm can define a norm ([4, 5]). Based on these facts and analogously to the definition of an *n*-norm, Nourouzi and Shabanian [10] defined a notion of *n*-modular spaces. In the present paper, we observe some topological properties of 2-modular spaces. We also introduce a definition of a 2-ρ-bounded 2-linear operator from a 2-modular space into a normed space. Furthermore, some properties of a 2-ρ-bounded 2-linear operator from a 2-modular space into a normed space are observed as well.

2. 2-modular Spaces

As usual, symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^*$ denote a natural numbers system, a real number system and an extended real numbers system, respectively. For any linear space *X*, $\dim(X)$ means the dimension of *X*. In this paper, we always assume that for any linear space *X*, the $\dim(X) \geq 2$, unless otherwise mentioned.

Further, we give a definition of a 2-modular, analogously with those of a 2-norm.

Definition 2.1. Let *X* be a real linear space with $\dim(X) \geq 2$. A real valued function $\rho(\cdot, \cdot) : X \times X \to \mathbb{R}^*$ is called a 2-*modular* on *X* if

(i) $\rho(x, y) = 0$ if and only if *x* and *y* are linearly dependent,

(ii) $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$,

(iii) $\rho(-x, y) = \rho(y, x)$ for every $x, y \in X$, and

(iv) $\rho(\alpha x + \beta y, z) \leq \rho(x, z) + \rho(y, z)$ for every $x, y, z \in X$ and for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If the condition (iv) is replaced by

(iv′) $\rho(\alpha x + \beta y,\ z) \leq \alpha\rho(x,\ z) + \beta\rho(y,\ z)$ for every $x,\ y,\ z \in X$ and for every $\alpha,\ \beta \geq 0$ with $\alpha + \beta = 1$,

then $\rho(\cdot,\ \cdot)$ is called a *convex* 2-*modular*.

It is easy to prove that $\rho(x,\ y) \geq 0$ for every $x,\ y \in X$. Moreover, following the condition (i) in Definition 2.1, we have

(i) $\rho(x,\ 0) = 0$ for every $x \in X$, and

(ii) if $\rho(x,\ y) = 0$ for every $y \in X$, then $x = 0$.

Following are examples of 2-modulars.

Example 2.2. Let $X = \mathbb{R}^2$. If the function $\rho : X \times X \to \mathbb{R}^*$ is defined by

$$\rho(x,\ y) = \text{abs}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right),$$

then ρ is a 2-modular on X.

Example 2.3. Let X be a real linear space and $\|\cdot,\ \cdot\|$ a 2-norm on X. Then

$$\rho(x,\ y) = \int_0^{\|x,\, y\|} (e^t - 1)\, dt$$

is a 2-modular on X.

It can be seen that every 2-norm on a linear space X is a 2-modular, but the converse is not true.

Example 2.4. Let $X = \mathbb{R}^2$. If the function $\rho : X \times X \to \mathbb{R}^*$ is defined by

$$\rho(x,\ y) = \sqrt{\text{abs}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right)},$$

then ρ is a 2-modular on X. However, ρ is not a 2-norm on X.

Theorem 2.5. *Any* 2-*modular on a real linear space X generates a modular on X.*

Proof. Let $\rho(\cdot, \cdot)$ be a 2-modular on a real linear space X. Take any linearly independent set of vectors $\{a_1, a_2\}$ on X. Define a function $\sigma : X \to \mathbb{R}^*$ by

$$\sigma(x) = \max\{\rho(x, a_1), \rho(x, a_2)\},$$

then $\sigma(-x) = \sigma(x)$ for every $x \in X$ and

$$\begin{aligned}\sigma(x) = 0 &\Leftrightarrow \max\{\rho(x, a_1), \rho(x, a_2)\} \\ &\Leftrightarrow \rho(x, a_1) = \rho(x, a_2) = 0 \\ &\Leftrightarrow \{x, a_1\} \text{ and } \{x, a_2\} \text{ are linearly dependent} \\ &\Leftrightarrow x = 0,\end{aligned}$$

since $\{a_1, a_2\}$ is linearly independent. Now, let $x, y \in X$ and $\alpha, \beta \geq 0$ be such that $\alpha + \beta = 1.$ Then

$$\begin{aligned}\sigma(\alpha x + \beta y) &= \max\{\rho(\alpha x + \beta y, a_1), \rho(\alpha x + \beta y, a_2)\} \\ &\leq \max\{\rho(x, a_1), \rho(x, a_2)\} + \max\{\rho(y, a_1), \rho(y, a_2)\} \\ &= \sigma(x) + \sigma(y).\end{aligned}$$

Thus, the function σ is a modular. □

The following theorem describes some basic properties of a 2-modular.

Theorem 2.6. *If* ρ *is a* 2-*modular on a real linear space X, then*

(i) $\rho(\lambda x, y) \leq \rho(x, y)$ *for every* $x, y \in X$ *and* $|\lambda| \leq 1.$

(ii) $\rho\left(\sum_{k=1}^{n} \lambda_k x_k, y\right) \leq \sum_{k=1}^{n} \rho(x_k, y)$ *for every* $x_k, \ y \in X$ *and* $\lambda_k \geq 0, k = 1, 2, ..., n,$ *with* $\sum_{k=1}^{n} \lambda_k = 1.$

(iii) $\rho(\alpha x, y) \le \rho(\beta x, y)$ *for every* $x, y \in X$ *and* $\alpha, \beta \in \mathbb{R}$ *with* $0 < \alpha \le \beta.$

Proof. (i) It is trivial for $\lambda = 0$ or $|\lambda| = 1.$ Now, let $0 < \lambda < 1.$ Then

$$\rho(\lambda x, y) = \rho(\lambda x + (1-\lambda)0, y) \le \rho(x, y).$$

Moreover, following the condition (iii) in Definition 2.1, then we have

$$\rho(\lambda x, y) \le \rho(x, y),$$

for every $-1 < \lambda < 0.$ So, (i) is proved.

(ii) We are going to prove (ii) by mathematical induction. It is true for x_1, x_2, y and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 = 1,$ because of the condition (iv) in Definition 2.1. Assume that it is true for $x_1, x_2, ..., x_n, y$ and $\lambda_1, \lambda_2, ..., \lambda_n$ ≥ 0 with $\sum_{k=1}^{n} \lambda_k = 1.$ Then

$$\rho\left(\sum_{k=1}^{n} \lambda_k x_k, y\right) \le \sum_{k=1}^{n} \rho(x_k, y).$$

Now, take any $x_1, x_2, ..., x_{n+1}, y \in X$ and $\lambda_1, \lambda_2, ..., \lambda_{n+1} \ge 0$ such that $\sum_{k=1}^{n+1} \lambda_k = 1,$ then there is a positive integer $j, 1 \le j \le n+1,$ such that $\lambda_j \ne 0.$ So, we have

$$\begin{aligned}\rho\left(\sum_{k=1}^{n+1} \lambda_k x_k, y\right) &= \rho\left((1-\lambda_j) \sum_{k=1, k\ne j}^{n+1} \frac{\lambda_k x_k}{1-\lambda_j} + \lambda_j x_j, y\right) \\ &\le \rho\left((1-\lambda_j) \sum_{k=1, k\ne j}^{n+1} \frac{\lambda_k x_k}{1-\lambda_j}, y\right) + \rho(x_j, y) \\ &\le \sum_{k=1, k\ne j}^{n+1} \rho(x_k, y) + \rho(x_j, y) = \sum_{k=1}^{n+1} \rho(x_k, y).\end{aligned}$$

(iii) Following condition (iv) in Definition 2.1, then the assertion follows. □

Let X be a real linear space. A 2-modular ρ on X is said to satisfy the Δ_2-*condition* if there exists a constant $K > 0$ such that $\rho(2x, y) \leq K\rho(x, y)$ for every $x, y \in X$. The 2-modular ρ as given in Example 2.4 satisfies the Δ_2-condition. However, the 2-modular ρ as given in Example 2.3 does not satisfy the Δ_2-condition.

Throughout this paper, we always assume that the 2-modular ρ satisfies the Δ_2-condition.

Let ρ be a 2-modular on a real linear space X. We define

$$X_\rho = \{x \in X : \rho(\lambda x, y) < \infty, \text{ for some } \lambda > 0 \text{ and for any } y \in X\}. \quad (2.1)$$

It can easily be proved that X_ρ is a real linear space. Moreover, X_ρ is a 2-modular space with respect to ρ. We can also prove that $\rho(x, y) < \infty$ for every $x \in X_\rho$ and for every $y \in X$.

Throughout this paper, X_ρ is always meant as given in (2.1).

3. Topological Properties of 2-modular Spaces

In this section, we introduce some topological concept with respect to a 2-modular. We begin our discussion by giving a notion of 2-modular convergent sequences in the space X_ρ.

Let X_ρ be a 2-modular space. A sequence $\{x_n\}$ in X_ρ is said to be 2-*modular convergent* (or ρ-*convergent*) to some $x \in X_\rho$, denoted by

$$\rho - \lim x_n = x,$$

if for every $y \in X_\rho$, $\lim \rho(x_n - x, y) = 0$, i.e., for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any integer $n \geq N$, we have $\rho(x_n - x, y) < \varepsilon$. In this case, the vector x is called a 2-*modular limit* (ρ-*limit*) of the sequence $\{x_n\}$.

Example 3.1. Let X and ρ be as given in Example 2.4. It is clear that $X_\rho = X$. Let $x_n = \left(\frac{1}{n}, 0\right)$ for every $n \in \mathbb{N}$ and $x = (0, 0)$. For any $y = (y_1, y_2) \in X$, we have

$$\rho(x_n - x, y) = \sqrt{\operatorname{abs}\left(\begin{vmatrix} \frac{1}{n} & 0 \\ y_1 & y_2 \end{vmatrix}\right)} = \sqrt{\frac{|y_2|}{n}}.$$

Given $\varepsilon > 0$, we can choose a positive integer N such that $\sqrt{\frac{|y_2|}{N}} < \varepsilon$. Hence, the sequence $\{x_n\}$ ρ-converges to x.

We observe some basic properties of the ρ-convergence of a sequence in any 2-modular space. Let us see the following theorems:

Theorem 3.2. *Let X_ρ be a 2-modular space and $\{x_n\}$ be a sequence in X_ρ. If $\{x_n\}$ is ρ-convergent, then its ρ-limit is unique.*

Proof. Since the 2-modular ρ satisfies the Δ_2-condition, there exists a constant $K > 0$ such that

$$\rho(2x, y) \le K\rho(x, y),$$

for every $x, y \in X_\rho$. Given any $\varepsilon > 0$. Suppose $\{x_n\}$ ρ-converges to x and z in X_ρ. For any $y \in X_\rho$, there exists an $N \in \mathbb{N}$ such that

$$\rho(x_N - x, y) < \frac{\varepsilon}{2K} \quad \text{and} \quad \rho(x_N - z, y) < \frac{\varepsilon}{2K}.$$

These imply

$$\rho(x - z, y) \le \rho(2(x_N - x, y)) + \rho(2(x_N - z, y)) < \varepsilon. \tag{3.1}$$

Since the expression (3.1) holds for any $\varepsilon > 0$, we obtain $\rho(x - z, y) = 0$ for every $y \in X_\rho$. This implies $x = z$. □

Theorem 3.3. *Let* X_ρ *be a* 2*-modular space and* $\{x_n\}$ *be a sequence in* X_ρ. *If for every* $z \in X_\rho$, $\lim \rho(x_n - x, z) = \lim \rho(x_n - y, z) = 0$ *for some* $x, y \in X_\rho$, *then*

(i) $\rho(\alpha x_n - \alpha x, z) = 0$ *for every real number* α, *and*

(ii) $\rho((x_n + y_n) - (x + y, z)) = 0.$

Proof. Since the 2-modular ρ satisfies the Δ_2-condition, there exists a constant $K > 0$ such that $\rho(2x, y) \le K\rho(x, y)$ for every $x, y \in X_\rho$.

(i) It is trivial for $\alpha = 0$. Let $\alpha > 0$ be an arbitrary, there is a positive integer p such that $\alpha < 2^p$. Given $\varepsilon > 0$. Since $\rho(x_n - x, z) = 0$, there exists an $N \in \mathbb{N}$ such that for every $n \ge N$, we have $\rho(x_n - x, z) < \dfrac{\varepsilon}{K^p}$. This implies

$$\rho(\alpha x_n - \alpha x, z) \le \rho(2^p(x_n - x), z) \le K^p \rho(x_n - x, z) < \varepsilon.$$

In other words, $\lim \rho(\alpha x_n - \alpha x, z) = 0$. Moreover, following the condition (iii) in Definition 2.1, we obtain $\lim \rho(\alpha x_n - \alpha x, z) = 0$ for every $\alpha \in \mathbb{R}$.

(ii) Since

$$\begin{aligned}\rho((x_n + y_n) - (x + y), z) &\le \rho(2(x_n - x), z) + \rho(2(y_n - y), z)\\ &\le K(\rho(x_n - x, z) + \rho(y_n - y, z)),\end{aligned}$$

the assertion follows. □

A sequence $\{x_n\}$ in X_ρ is called a ρ-*Cauchy sequence* if for every $\varepsilon > 0$, there is a positive integer *N* such that

$$\rho(x_n - x_m, y) < \varepsilon,$$

for every $m, n \ge N$. The correlation between ρ-convergent and ρ-Cauchy sequences is formulated in the following theorem:

Theorem 3.4. *Every* ρ-*convergent sequence in* X_ρ *is a* ρ-*Cauchy sequence*.

Proof. We can choose a constant $K > 0$ such that $\rho(2x, y) \le K\rho(x, y)$ for all $x, y \in X_\rho$, since the 2-modular ρ satisfies the Δ_2-condition. Now, let $\{x_n\}$ be any sequence in X_ρ that ρ-converges, say to some $x \in X_\rho$. Given any $\varepsilon > 0$ and $y \in X_{rho}$, then there is a positive integer N such that $\rho(x_n - x, y) < \frac{\varepsilon}{3K}$ for every $n \ge N$. Further, for any $m, n \ge N$, we have

$$\begin{aligned}\rho(x_n - x_m, y) &\le \rho(2(x_n - x), y) + \rho(2(x - x_m), y)\\ &\le K(\rho(x_n - x, y) + \rho(x_m - x, y)) < \varepsilon.\end{aligned}$$

So, the proof is complete. □

We also characterize ρ-Cauchy sequences, as given in the following theorem:

Theorem 3.5. *A sequence* $\{x_n\}$ *in* X_ρ *is* ρ-*Cauchy if and only if* $\{\alpha x_n\}$ *is a* ρ-*Cauchy sequence for all* $\alpha \in \mathbb{R}$.

Proof. ($\Leftarrow$:) By taking $\alpha = 1$, the assertion follows.

($\Rightarrow$:) It is trivial for $\alpha = 0$. Let $\alpha > 0$ be an arbitrary. Then there is a positive integer p such that $\alpha < 2^p$. Since the 2-modular ρ satisfies the Δ_2-condition, there is a constant $K > 0$ such that $\rho(2x, y) \le K\rho(x, y)$ for all $x, y \in X_\rho$.

Let $\{x_n\}$ be a ρ-Cauchy sequence. Given $\varepsilon > 0$ and $y \in X_\rho$, there exists an $N \in \mathbb{N}$ such that for every $m, n \ge N$, we have $\rho(x_n - x_m, y) < \frac{\varepsilon}{K^p}$. This implies

$$\rho(\alpha x_n - \alpha x_m, y) \le \rho(2^p(x_n - x_m), y) \le K^p\rho(x_n - x_m, y) < \varepsilon.$$

In other words, $\{\alpha x_n\}$ is a ρ-Cauchy sequence. Moreover, following the condition (iii) in Definition 2.1, we obtain $\{\alpha x_n\}$ is a ρ-Cauchy sequence for every $\alpha \in \mathbb{R}$. □

4. 2-linear Operators

Let X be a real linear space. A notation X^2 is meant $X \times X$. The following definition refers to [1, 11].

Definition 4.1. Let X and Y be real linear spaces. An operator $T : X^2 \to Y$ is said to be 2-*linear* if for every $x, y, u, v \in X$ and $\alpha, \beta \in \mathbb{R}$, the following conditions hold:

(i) $T(x + y, u + v) = T(x, u) + T(x, v) + T(y, u) + T(y, v)$.

(ii) $T(\alpha x, \beta y) = \alpha\beta T(x, y)$.

Analogous to the definition of a 2-bounded 2-linear operator on 2-norm spaces, we define a 2-ρ-bounded 2-linear operator on 2-modular spaces. Let X_ρ be a 2-modular space and Y be a normed space. A 2-linear operator $T : X_\rho^2 \to Y$ is said to be 2-ρ-*bounded* if there exists a real constant $M > 0$ such that

$$\| T(x, y) \| \leq M\rho(x, y),$$

for every $x, y \in X_\rho$. Let us consider the following example.

Example 4.2. Let X and ρ be as given in Example 2.2. Note that $X_\rho = X$. If an operator $T : X_\rho^2 \to \mathbb{R}$ is defined by

$$T(x, y) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \quad x, y \in X_\rho,$$

then we can show that T is a 2-linear operator. Moreover, since

$$|T(x, y)| = \text{abs}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) = \rho(x, y)$$

for every $x, y \in X_\rho$, T is 2-ρ-bounded.

Let X_ρ be a 2-modular space and Y be a normed space. If $T : X_\rho^2 \to Y$ is a 2-ρ-bounded linear operator, then it is easy to prove that $T(x, y) = 0$ for every $x, y \in X_\rho$ which are linearly dependent. The collection of all 2-ρ-bounded linear operators $T : X_\rho^2 \to Y$ will be denoted by $B(X_\rho^2, Y)$. It is easy to check that $B(X_\rho^2, Y)$ is a real linear space. Moreover, one can define a function $\sigma : B(X_\rho^2, Y) \to \mathbb{R}^*$ by

$$\sigma(T) = \sup\left\{ \frac{\| T(x, y) \|}{\rho(x, y)} : x, y \in X_\rho, \rho(x, y) \neq 0 \right\}. \tag{4.1}$$

The theorem below shows that the function σ as given in (4.1) is a modular.

Theorem 4.3. *The function* $\sigma : B(X_\rho^2, Y) \to \mathbb{R}^*$ *as given in* (4.1) *is a modular on* $B(X_\rho^2, Y)$.

Proof. (i) If $T = 0$, then the definition of σ is obviously followed by $\sigma(T) = 0$. Conversely, if $\sigma(T) = 0$, then $T(x, y) = 0$ for all $x, y \in X_\rho$ which are not linearly dependent. Since $T(x, y) = 0$ for every $x, y \in X_\rho$ which are linearly dependent, we get $T(x, y) = 0$ for every $x, y \in X_\rho$. Hence, $T = 0$.

(ii) It is clear that $\sigma(-T) = \sigma(T)$ for every $T \in B(X_\rho^2, Y)$.

(iii) Take any $S, T \in B(X_\rho^2, Y)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then

$$\sigma(\alpha S + \beta T) = \sup\left\{\frac{\| \alpha S(x, y) + \beta T(x, y) \|}{\rho(x, y)} : \rho(x, y) \neq 0, x, y \in X_\rho\right\}$$

$$\leq |\alpha| \sup\left\{\frac{\| S(x, y) \|}{\rho(x, y)} : \rho(x, y) \neq 0, x, y \in X_\rho\right\}$$

$$+ |\beta| \sup\left\{\frac{\| T(x, y) \|}{\rho(x, y)} : \rho(x, y) \neq 0, x, y \in X_\rho\right\}$$

$$\leq \sigma(S) + \sigma(T).$$

From (i), (ii) and (iii), the assertion follows. □

The following theorem states necessary and sufficient conditions so that a 2-linear operator from a 2-modular space into a normed space is 2-ρ-bounded.

Theorem 4.4. *Let* X_ρ *be a* 2-*modular space and* Y *be a normed space. A* 2-*linear operator* $T : X_\rho^2 \to Y$ *is* 2-ρ-*bounded if and only if there is a constant* $M > 0$ *such that*

$$\| T(x, y) - T(u, v) \| \leq M\{\rho(x - u, y) + \rho(u, y - v)\}$$

and

$$\| T(x, y) - T(u, v) \| \leq M\{\rho(x - u, v) + \rho(x, y - v)\}$$

for all $x, y, u, v \in X_\rho$.

Proof. (⇒:) Since T is 2-ρ-bounded, there exists a real constant $M > 0$ such that

$$\| T(x, y) \| \leq M\rho(x, y),$$

for every $x, y \in X_\rho$. Take any $x, y, u, v \in X_\rho$, we have

$$\| T(x, y) - T(u, v) \| = \| T(x - u, y) - T(u, y - v) \|$$

$$\leq M\{\rho(x - u, y) + \rho(u, y - v)\}$$

and

$$\| T(x, y) - T(u, v) \| = \| T(x - u, v) - T(x, y - v) \|$$
$$\leq M\{\rho(x - u, v) + \rho(x, y - v)\}.$$

(⇐:) It is obvious. □

Theorem 4.5. *Let* X_ρ *be a* 2*-modular space and* Y *be a normed space. If for any* 2*-linear operator* $T : X_\rho^2 \to Y$, $\sigma(T)$ *is as defined in* (4.1), *then*

$$\sigma(T) = \inf\{M > 0 : \| T(x, y) \| \leq M\rho(x, y), x, y \in X_\rho\}.$$

Proof. Since $\| T(x, y) \| \leq \sigma(T)\rho(x, y)$ for every $x, y \in X_\rho$,

$$\inf\{M > 0 : \| T(x, y) \| \leq M\rho(x, y), x, y \in X_\rho\} \leq \sigma(T).$$

Conversely, if $K = \inf\{M > 0 : \| T(x, y) \| \leq M\rho(x, y), x, y \in X_\rho\}$, then

$$\frac{\| T(x, y) \|}{\rho(x, y)} \leq K$$

for every $x, y \in X_\rho$ with $\rho(x, y) \neq 0$. Hence, $\sigma(T) \leq K$. □

Let X_ρ be a 2-modular space and Y be a normed space. An operator $T : X_\rho^2 \to Y$ is said to be (n, ρ)-*continuous* at $(x_0, y_0) \in X_\rho^2$ if for every real number $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X_\rho^2$ with

(i) $\rho(x_0 - x, y_0) < \delta$ and $\rho(x, y - y_0) < \delta$, or

(ii) $\rho(x_0 - x, y) < \delta$ and $\rho(x_0, y - y_0) < \delta$,

we have $\| T(x, y) - T(x_0, y_0) \| < \varepsilon$. The operator T is said to be (n, ρ)-*continuous* on $E \subset X_\rho^2$ if it is (n, ρ)-continuous at every $(x, y) \in E$. And T is said to be (n, ρ)-*continuous* if it is (n, ρ)-continuous on X_ρ^2.

Example 4.6. Let $X, \rho, X_\rho,$ and $T : X_\rho^2 \to \mathbb{R}$ be as given in Example 4.2. Take any $(x_0, y_0) \in X_\rho^2$. For any $(x, y) \in X_\rho^2$, we have

$$|T(x, y) - T(x_0, y_0)| \leq \rho(x - x_0, y_0) + \rho(x, y - y_0)$$

and

$$|T(x, y) - T(x_0, y_0)| \leq \rho(x - x_0, y) + \rho(x_0, y - y_0).$$

Thus, T is (n, ρ)-continuous at (x_0, y_0).

Theorem 4.7. *Let X_ρ be a 2-modular space and Y be a normed space. If a 2-linear operator $T : X_\rho^2 \to Y$ is 2-ρ-bounded, then it is (n, ρ)-continuous.*

Proof. By Theorem 4.4, the assertion follows. □

By adding the convex property to the 2-modular ρ, we can prove the equivalence between 2-ρ-boundedness and (n, ρ)-continuity of a 2-linear operator $T : X_\rho^2 \to Y$. For proving this, we need the following lemma:

Lemma 4.8. *Let X_ρ be a 2-modular space and Y be a normed space. A 2-linear operator $T : X_\rho^2 \to Y$ is (n, ρ)-continuous at $(0, 0) \in X_\rho^2$ if and only if for any sequence $\{(x_n, y_n)\}$ that satisfies $\lim \rho(x_n, y_n) = 0$, we have $\lim \| T(x_n, y_n) \| = 0$.*

Proof. The proof is standard, so it is omitted. □

Theorem 4.9. *Let X_ρ be a 2-modular space with ρ be convex, Y be a normed space, and $T : X_\rho^2 \to Y$ be a 2-linear operator. The following statements are equivalent:*

(i) *The operator T is* (n, ρ)-*continuous*.

(ii) *The operator T is* (n, ρ)-*continuous at* $(0, 0)$.

(iii) *The set* $\{\| T(x, y) \| : \rho(x, y) \leq 1\}$ *is bounded*.

(iv) *The operator T is* 2-ρ-*bounded*.

Proof. (i) ⇒ (ii) is obvious. (iv) ⇒ (i) follows from Theorem 4.7. What remains to show are (ii) ⇒ (iii) and (iii) ⇒ (iv).

(ii) ⇒ (iii) Suppose the set $\{\| T(x, y) \| : \rho(x, y) \leq 1\}$ is unbounded. Then for every $n \in \mathbb{N}$, there exists $(x_n, y_n) \in X_\rho^2$ such that $\rho(x_n, y_n) \leq 1$, but

$$\| T(x_n, y_n) \| \geq n^2.$$

Set $u_n = \frac{x_n}{n}$ and $v_n = \frac{y_n}{n}$, then

$$\rho(u_n, v_n) \leq \frac{1}{n^2} \rho(x_n, y_n) \leq \frac{1}{n^2}.$$

This follows from the convexity of ρ. So, $\lim \rho(u_n, v_n) = 0$. By Lemma 4.8, it must be $\lim \| T(x_n, y_n) \| = 0$. However, it is impossible because

$$\| T(u_n, v_n) \| = \frac{1}{n^2} \| T(x_n, y_n) \| \geq 1.$$

So, $\{\| T(x, y) \| : \rho(x, y) \leq 1\}$ is bounded.

(iii) ⇒ (iv) By the hypothesis, there exists $M > 0$ such that $\| T(x, y) \| \leq M$, whenever $\rho(x, y) \leq 1$.

Take any $(x, y) \in X_\rho^2$. It is trivial if $\rho(x, y) \leq 1$. If $\rho(x, y) > 1$, then by the convexity of ρ,

$$\rho\left(\frac{(x, y)}{\rho(x, y)}\right) \leq 1.$$

Hence,

$$\| T(x, y) \| \leq M\rho(x, y)$$

and the proof is finished. □

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References

[1] H. Y. Chu, S. H. Ku and D. S. Kang, Characterizations on 2-isometries, J. Math. Anal. Appl. 340 (2008), 641-628.

[2] S. Gähler, Lineare 2-normierte Räume, Math. Nachr. 28 (1964), 1-43.

[3] S. Gähler, Untersuchungen über verallgemenerte *m*-metrische Räume, I, II, III, Math. Nachr. 40 (1969), 165-189.

[4] H. Gunawan and M. Mashadi, On *n*-normed spaces, Int. J. Math. Math. Sci. 27 (2001), 631-639.

[5] H. Gunawan and Mashadi, On finite-dimensional *2*-normed spaces, Soochow J. Math. 27(3) (2011), 321-329.

[6] M. A. Khamsi, A convexity property in modular function spaces, Department of Mathematical Sciences, The University of Texas at El Paso, 1980.

[7] S. Mazur and W. Orlicz, On some classes of linear spaces, Studia Math. 17 (1958), 97-119.

[8] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Math., Vol. 1034, Springer-Verlag, 1983.

[9] J. Musielak and W. Orlicz, On modular spaces, Studia Math. 18 (1959), 49-65.

[10] K. Nourouzi and S. Shabanian, Operator defined on *n*-modular spaces, Mediterr. J. Math. 6 (2009), 431-446.

[11] N. Srivastava, S. Bhattacharya and S. N. Lal, On Hahn-Banach extension of linear *n*-functionals in *n*-normed spaces, Math. Maced. 4 (2006), 25-32.

[12] Supama, On some common fixed point theorems in modulared spaces, Int. Math. Forum 7(52) (2012), 2571-2579.