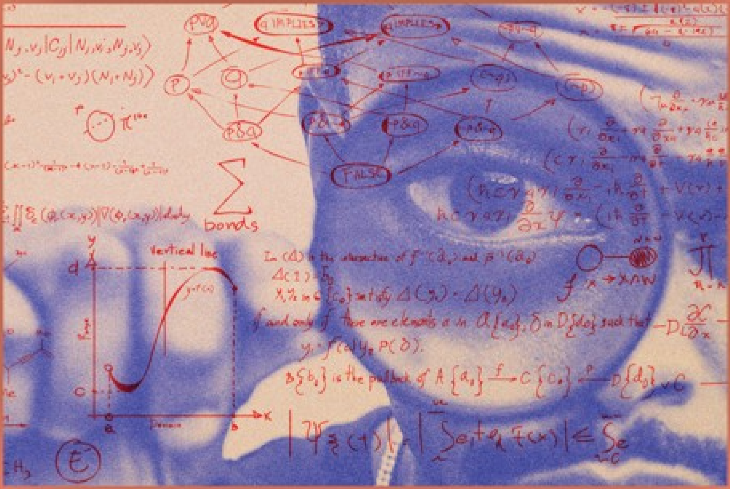
DIKTAT MATA KULIAH

KALKULUS



Dr. Ir. Siti Jamilatun, MT.

K-Media

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Dr. Ir. Siti Jamilatun, MT.

Penerbit K-Media
Yogyakarta, 2022

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KATA PENGANTAR

بسم الله الرحمن الرحيم

Puji syukur penulis ucapkan ke hadapan Allah SWT karena berkat rahmat, hidayah, dan karuniaNyalah penulis dapat menyelesaikan Diktat untuk Mata Kuliah **Kalkulus** ini.

Tujuan penulisan diktat kuliah ini adalah untuk memberikan pemahaman dan pembahasan dalam mendalami kalkulus dengan Latihan soal dan pembahasan tentang Integral, Diferensial, Fungsi Gama dan Beta, Deret, Limit Fungsi, dan Deret Pangkat. Oleh karena itu, diharapkan setelah membaca dan mempelajari diktat ini mahasiswa dapat menyelesaikan soal-soal kalkulus dengan baik.

Ucapan terima kasih penulis sampaikan kepada semua pihak yang telah membantu penulis dalam menyelesaikan penulisan diktat ini. Akhirnya sumbang saran sangat penulis harapkan demi kesempurnaan diktat ini. Semoga diktat ini bermanfaat bagi seluruh pembaca, khususnya mahasiswa.

Yogyakarta, Desember 2021

Penulis

DAFTAR ISI

BAB I
INTEGRAL

Integral adalah antideferensial (antiturunan) atau operasi invers terhadap diferensial. Menentukan fungsi *f(x)* dari *f'(x)* berarti menentukan antiturunan dari *f'(x)*.

Integral tak tentu

Jika F(x) adalah fungsi dari turunan F'(x) = f(x),maka f(x) adalah integral.

$$\int f(x)dx$$

$\int 3x^2\, dx = x^3 + c$, dimana c = koefisien

(x^3); $(x^3 + 5)$; $(x^3 - 4)$ diturunkan / dideferensialkan menjadi

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 + 5\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 - 4\right) = 3x^2$$

I.1 Integral Baku

A. Rumus – Rumus Integrasi Baku

1. $\int \frac{d}{dx} f(x)\, dx = f(x) + c$	6. $\int a^u\, du = \frac{u}{\ln a} + c;\ a > 0;\ a \neq 1$
2. $\int (u \pm v)\, dx = \int u dx \pm \int v\, dx$	7. $\int e^u\, du = e^u + c$
3. $\int k.\, u.\, dx = k \int u dx$, k = konstanta	8. $\int \sin u\, du = -\cos u + c$
4. $\int u^m\, du = \frac{1}{m+1} u^{m+1} + c$;	9. $\int \cos u\, du = \sin u + c$
5. $\int \frac{du}{u} = \ln \lvert u \rvert + c$	10. $\int tg\, u\, du = \ln \lvert \sec u \rvert + c$

11.	$\int \text{cotg u du} = \ln|\sin u| + c$
12.	$\int \sec u \, du = \ln|\sec u + \text{tg } u| + c$
13.	$\int \text{cosec u du} = \ln|\text{cosec } 4 - \text{tg } u| + c$
14.	$\int \sec^2 u \, du = \tan u + c$
15.	$\int \text{cosec u cotan u du} = - \text{cosec u} + c$
16.	$\int \sec u \tan u \, du = \sec u + c$
17.	$\int \text{cosec u cotan u du} = - \text{cosec u} + c$
18.	$\int \frac{du}{a^2+u^2} = \frac{1}{a} \text{arc tan} \frac{u}{a} + c$
19.	$\int \frac{du}{\sqrt{a^2-u^2}} = \text{arc sin} \frac{u}{a} + c$
20.	$\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a} \text{arc sec} \frac{u}{a} + c$
21.	$\int \frac{du}{u^2+a^2} = \frac{1}{2a} \ln \left\|\frac{u-a}{u+a}\right\| + c$
22.	$\int \frac{du}{\sqrt{u^2+a^2}} = \ln \|u\| + \sqrt{u^2+a^2}\| + c$
23.	$\int \frac{du}{\sqrt{u^2-a^2}} = \ln \|u\| + \sqrt{u^2-a^2}\| + c$
24.	$\int \sqrt{a^2-u^2} du = \frac{1}{2} u\sqrt{a^2-u^2} + \frac{1}{2} a^2 \text{ arc sin} \frac{u}{a} + c$
25.	$\int \sqrt{u^2+a^2} du = \frac{1}{2} u\sqrt{u^2+a^2} + \frac{1}{2} a^2 \ln \|u + \sqrt{u^2+a^2}\| + c$
26.	$\int \sqrt{u^2-a^2} \, du = \frac{1}{2} u\sqrt{u^2-a^2} - \frac{1}{2} a^2 \ln \|u + \sqrt{u^2-a^2}\| + c$

B. Rumus Integrasi Trigonometri Baku

1.	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + C$ (asalkan $n \neq -1$)
2.	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\therefore \int \frac{1}{x} dx = \ln x + C$
3.	$\frac{d}{dx}(e^x) = e^x$	$\therefore \int e^x \, dx = e^x + C$
4.	$\frac{d}{dx}(e^{kx}) = ke^{kx}$	$\therefore \int e^{kx} \, dx = \frac{e^{kx}}{k} + C$
5.	$\frac{d}{dx}(a^x) = a^x \ln a$	$\therefore \int a^x \, dx = \frac{a^x}{\ln a} + C$
6.	$\frac{d}{dx}(\cos x) = - \sin x$	$\therefore \int \sin x \, dx = -\cos x + C$
7.	$\frac{d}{dx}(\sin x) = \cos x$	$\therefore \int \cos x \, dx = \sin x + C$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$	$\therefore \int \sec^2 x \, dx = \tan x + C$
9. $\frac{d}{dx}(\cosh x) = \sinh x$	$\therefore \int \sinh x \, dx = \cosh x + C$
10. $\frac{d}{dx}(\sinh x) = \cosh x$	$\therefore \int \cosh x \, dx = \sinh x + C$
11. $\frac{d}{dx}\log a^x = \frac{1}{x \ln a}$	$\therefore \int \frac{1}{x \ln a} \, dx = \log a^x + C$
12. $\frac{d}{dx}(\sin^{-1} x) = \frac{-1}{\sqrt{(1-x^2)}}$	$\therefore \int \frac{1}{\sqrt{(1-x^2)}} dx = \sin^{-1} x + C$
13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{(1-x^2)}}$	$\therefore \int \frac{-1}{\sqrt{(1-x^2)}} dx = \cos^{-1} x + C$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\therefore \int \frac{1}{\sqrt{(1+x^2)}} dx = \tan^{-1} x + C$
15. $\frac{d}{dx}(\sinh^{-1}) = \frac{1}{\sqrt{x^2+1}}$	$\therefore \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C$
16. $\frac{d}{dx}(\cosh^{-1}) = \frac{1}{\sqrt{x^2-1}}$	$\therefore \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$
17. $\frac{d}{dx}(\tanh^{-1}) = \frac{1}{1-x^2}$	$\therefore \int \frac{1}{1-x^2} dx = \tanh^{-1} x + C$

C. Contoh

Urutan nomor sesuai rumus

$$A.5 \rightarrow \int \frac{du}{u} = \ln u + C$$

a. $\int \frac{dx}{2+3x} = \frac{1}{3} \int \frac{d(2+3x)}{2+3x} = \frac{1}{3} \ln(2 + 3x) + C$

b. $\int \frac{x^2 dx}{x^3-3} = \frac{1}{3} \int \frac{d(x^3-3)}{x^3-3} = \frac{1}{3} \ln(x^3 - 3x) + C$

$x^2 \, dx = d(x^3\text{-}3)$

$x^2 = (3x^2) \cdot \frac{1}{3}$

$x^2 = x^2$

c. $\int \frac{x \, dx}{x^2-1} = \frac{1}{2} \int \frac{d(x^2-1)}{x^2-1} = \frac{1}{2} \ln(x^2 - 1) + C$

$$A.6 \rightarrow \int a^u \, du = \frac{a^u}{\ln a} + C$$

a. $\int 10^x\,dx = \frac{10^x}{\ln 10} + C$

A.7 → $\int e^u\,du = e^u + C$

a. $\int 6\,e^{3x}\,dx = \frac{1}{3}\int 6\,e^{3x}\,d\,(3x) = 2\,e^{3x} + C$

b. $\int \frac{1}{3} e^{10x}\,dx = \frac{1}{3}.\frac{1}{10}\int e^{10x}\,d\,(10x) = \frac{1}{30} e^{10x} + C$

c. $\int 5\,e^{ax}\,dx = 5\frac{1}{a}\int e^{ax}\,d\,(ax) = \frac{5}{a} e^{ax} + C$

A.8 → A.17

a. $\int \sin\frac{1}{2}x\,dx = 2\int \sin\frac{1}{2}x\,d\,(\frac{1}{2}x)$

$= -2\cos\frac{1}{2}x + C$

b. $\int \sin 4x\,dx = \frac{1}{4}\int \sin 4x\,d\,(4x)$

$= -\frac{1}{4}\cos\,(4x) + C$

c. $\int tg\,2x\,dx = \int \frac{\sin 2x}{\cos 2x} dx$

$= \frac{1}{2}\int \frac{\sin 2x}{\cos 2x} d\,(2x)$

$= -\frac{1}{2}\int \frac{d(\cos 2x)}{\cos 2x} = -\frac{1}{2}\ln|\sec 2x| + C$

$= -\frac{1}{2}\int \frac{du}{u} \quad = -\frac{1}{2}\ln|\cos 2x| + C$

atau

$\int tg\,x\,dx = \ln|\sec x| + C$

$\int tg\,2x\,dx = \frac{1}{2}\int tg\,2x\,d\,(2x)$

$= \frac{1}{2}\ln|\sec 2x| + C$

d. $\int \frac{\sin\sqrt{x}\,dx}{\sqrt{x}} = \int \sin\sqrt{x}\;x^{-\frac{1}{2}}dx$

$= 2\int \sin\sqrt{x}\,d\,(\sqrt{x})$

$= -2\cos\sqrt{x} + C$

A. 18 → $\int \frac{du}{a^2+u^2} = \frac{1}{a}\,arc\,tg\,\frac{u}{a} + C$

2.6 $\int a^4\,du = \frac{a^4}{\ln a} + C$

a. $\int \frac{dx}{x^2+9} = \frac{dx}{x^2+3^2} = \frac{1}{3}\,\text{arc tg}\,\frac{x}{3} + C$

b. $\int \frac{dx}{x^2-4+13} \quad = \quad \int \frac{dx}{(x-2)^2+9}$ yang mengandung unsur x = u

↓

$(x^2\text{-}4x+4)$

$= \int \frac{d(x-2)}{(x-2)^2+3^2}$

$= \frac{1}{3}\,\text{arc tg}\,\frac{x-2}{3} + C$

$$A.19 \rightarrow \int \frac{du}{\sqrt{a^2-u^2}} = \text{arc sin}\,\frac{u}{a} + C$$

a. $\int \frac{dx}{\sqrt{25-x^2}} = \frac{dx}{\sqrt{5^2-x^2}} = \text{arc sin}\,\frac{x}{5} + C$

$$A.20 \rightarrow \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{2}\,\text{arc sec}\,\frac{u}{a} + C$$

a. $\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{dx}{x\sqrt{x^2-3^2}} = \frac{1}{2}\text{arc sec}\,\frac{x}{3} + C$

A.21→ A.26

a. $\int \frac{dx}{x^2-4} \quad = \int \frac{dx}{u^2-2^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

$= \frac{1}{2.2}\ln\left|\frac{x-2}{x+2}\right| + C$

$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

b. $\int \frac{dx}{9x^2-4} \quad = \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

$\int \frac{du}{u^2-a^2} \quad = \frac{1}{3}\int \frac{d(3x)}{(3x)^2-2^2}$

$\int \frac{du}{u^2-a^2} \quad = \frac{1}{2.2}.\frac{1}{3}\ln\left|\frac{x-2}{x+2}\right| + C$

$= \frac{1}{12}\ln\left|\frac{x-2}{x+2}\right| + C$

c. $\sqrt{1-4x^2}\,dx \rightarrow \sqrt{a^2-u^2}du$

$= \frac{1}{2}\,u\,\sqrt{a^2-u^2} + \frac{1}{2}a^2\,\text{arc sin}\,\frac{u}{a} + C$

d. $\sqrt{1^2-(2x)^2}du$ $= \frac{1}{2}\, 2x\sqrt{1^2-(2x)^2} + \frac{1}{2}1^2 \text{ arc sin } \frac{2x}{1} + C$

$= \frac{2x}{2}\sqrt{1-4x^2} + \frac{1}{2}\text{arc sin } 2x + C$

$= x\sqrt{1-4x^2} + \frac{1}{2}\text{arc sin } 2x + C$

I.2 Fungsi Dari Suatu Fungsi Linier Dalam X

Contoh :

1. $\int (5x\text{-}2)^6\, dx$

misal $5x - 2 = z$

$5x = z + 2$

$x = \frac{z+2}{5}$

$dx = \frac{1}{5}\, dz$

Jadi $\int (5x\text{-}2)^6\, dx = \int (z)^6\, dz$

$= \frac{1}{5} \cdot \frac{1}{7} \cdot z^7 + C$

$= \frac{1}{35} \cdot z^7 + C$

$= \frac{1}{35}(5x - 2)^7 + C$

2. $\int e^{5x+4}\, dx$

misal $z = 5x + 4$

$\frac{dz}{dx} = 5$

sehingga $\frac{dx}{dz} = \frac{1}{5}$

Persamaan integralnya menjadi:

$$\int e^z\, dx = \int e^z \frac{dx}{dz} dz = \int e^z \frac{1}{5} dz = \frac{1}{5} e^{5x+4} + C$$

3. $\int \cos\,(7x+2)\, dx$

misal $z = 7x + 2$

$dz = 7\, dx$

$dx = \frac{1}{7}\, dz$

Persamaan integralnya menjadi:

$$\int \cos z\; dx = \int \cos z\, \frac{1}{7}\, dz = \frac{1}{7}\sin z = \frac{1}{7}\sin\,(7x+2) + C$$

I.3 Integral Dalam Bentuk Pembagian 2 Fungsi

$$\int \frac{f'(x)}{f(x)}\,dx$$

Jika pembilang dari suatu integral merupakan turunan dari penyebut maka akan menjadi :

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$$

Contoh :

1. $\int \frac{2x-3}{x^2-3x+5}dx$ → misal : $z = x^2 - 3x + 5$

   $dz = (2x - 3)\,dx$

   $= \int \frac{dz}{z}$

   $= \ln z + C$

   $= \ln (x^2 - 3x + 5) + C$

2. $\int \frac{4x-4}{2x^2-4x+5}dx$ → misal : $z = 2x^2 - 4x + 5$

   $dz = (4x - 4)\,dx$

   $= \int \frac{dz}{z}$

   $= \ln z + C$

   $= \ln (2x^2 - 4x + 5) + C$

3. $\int \frac{6x^2}{x^3-4}\,dx$ → misal : $z = x^3 - 4$

   $dz = 3x^2\,dx$

   $$2\int \frac{3x^2}{x^3-4}\,dx = 2\ln(x^3-4) + C$$

4. $\int \cot x\,dx$

   $$\textit{dibuat bentuk} \int \frac{f'(x)}{f(x)}\,dx \quad \textit{sehingga menjadi} \int \frac{cos}{sin}\,dx$$

   $$\int \cot x\,dx = \ln \sin x + C$$

I.5 Integral Dalam Bentuk Perkalian 2 Fungsi

$$\int f(x) \,.\, f'(x)\ dx$$

Jika pengali dari suatu integral merupakan turunan dari pengali yang lain maka akan menjadi :

$$\int f(x) \,.\, f'(x)\ dx = \frac{1}{2}\ f(x)^2 + C$$

1. $\int \tan x \,.\, \sec^2 x\ dx \rightarrow$ misal : $z = tg\ x$

   $dz = \sec^2 dx$

   $= \int z\ dz$

   $= \frac{1}{2} z^2 + C$

   $= \frac{1}{2}(tg)^2 + C$

2. $\int \frac{\ln x}{x}\ dx$

   $$\int \frac{\ln x}{x}\ dx = \int \ln x \,.\, \frac{1}{x}\ dx$$

   $$= \int \ln x\ \,.\, d(\ln x)$$

   $$= \frac{(\ln x)^2}{2} + C$$

3. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx$

   $$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx = \int \sin^{-1} x \,.\, \frac{1}{\sqrt{1-x^2}}\ dx$$

   $$= \int \sin^{-1} x \,.\, d(\sin^{-1} x)$$

   $$= \frac{(\sin^{-1} x)^2}{2} + C$$

I.6 Integral Parsial

$$u.v = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$$

$$\boxed{\int u\, dv = u\,.\,v - \int v\, du}$$

Penting!

Cara menentukan"u"

- Jika f (x) → logaritma, maka yang dipilih sebagai "u" adalah yang ada logaritmanya.
  Contoh → u = ln x
- Jika f (x) → tidak ada faktor log, maka yang dipilih sebagai "u" yang mempunyai faktor → x
  Contoh → $u = x^2$
  u = x
- Jika f (x) → tidak ada log (ln) dan pangkat x, maka yang dipilih sebagai "u" adalah e = u

Contoh :

1. $\int x^2 \ln x\, dx$

   coba

   (1) $x^2 = u$

   $\ln x = dv \rightarrow$ maka $\int \ln x\, dx$ = X tidak ada dalam daftar integral baku

   (2) $\ln x = u$

   $x^2 = dv \rightarrow$ maka $\int x^2\, dx$

   $v = \frac{1}{3}x^3 \rightarrow$ ada dalam daftar

   $\int u\, dv = u\,.\,v - \int v\, du$

   $\int \ln x\, x^2 = \ln x\, \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \,.\, \frac{1}{x}dx$

   $= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$

   ↓

   $(x^3 \cdot x^{-1})$

   $= \frac{1}{3}x^3 \ln x - \frac{1}{3}\,.\,\frac{1}{3}x^3 + C$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

2. $\int x^2\, e^{3x}\, dx$

   mis : $u = x^2 \rightarrow du = 2x\, dx$

   $dv = e \rightarrow \frac{1}{3}e^{3x}$

   maka :

   $\int x^2\, e 3x\, dx \quad = \quad x^2 \left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x} \cdot 2x\, dx$

   $= \quad \frac{x^2}{3}e^{3x} - \frac{2}{3}\int x \cdot e^{3x}\, dx$

   $= \quad \frac{x^2}{3}e^{3x} - \frac{2}{3}x\, \frac{1}{3}e^{3x} + \int \frac{1}{3}e^{3x}\, dx$

   Mis : $u = x \rightarrow du = 1\, dx$

   $dv = e^{3x}$

   $v = \frac{1}{3}e^{3x}$

   $= \quad \frac{x^2 . e^{3x}}{3} - \frac{2}{9}x\, e^{3x} + \frac{1}{3} \cdot \frac{1}{3}e^{3x} + C$

   $= \quad \frac{x^2 . e^{3x}}{3} - \frac{2}{9}x\, e^{3x} + \frac{1}{9}e^{3x} + C$

3. $\int e^{3x} \sin x\, dx$

   mis : $u = e^{3x} \rightarrow du = 3\, e^{3x}\, dx$

   $dv = \sin x \rightarrow \quad \int \sin x \, . \, dx = - \cos x = v$

   $\int e^{3x} \sin x\, dx \quad = \quad uv - \int v\, du$

   $= \quad e^{3x} (-\cos x) - \int (-\cos x) \, . \, 3\, e^{3x}\, dx$

   $= \quad -e^{3x} \cos x + 3\, [e^{3x} \sin x - 3 \int \sin x \, . \, e^{3x}\, dx]$

   $\int e^{3x} \sin dx = - e^{3x} \cos x + 3\, e^{3x} - \sin x - 9 \int e^{3x} \sin x\, dx$

   $\int e^{3x} \sin dx + 9 \int e^{3x} \sin x\, dx = -e^{3x} \cos x + 3\, e^{3x} \sin x$

   $10 \int e^{3x} \sin x\, dx = -e^{3x} \cos x + 3\, e^{3x} - \sin x$

   $\int e^{3x} \sin x\, dx = \frac{1}{10} (-e^{3x} \cos x + 3\, e^{3x} \sin x)$

I.6 Integrasi Dengan Pecahan Parsial

Misal : $\int \frac{x+1}{x^2-3x+2}$ dx → tidak ada dalam bentuk baku

Nyatakan dalam pecahan parsial!

$\frac{x+1}{x^2-3x+2} \rightarrow \frac{3}{x-2} - \frac{2}{x-1} \rightarrow$ dari mana ?

Kaidah Pecahan Parsial

a. Pembilang dari fungsi lebih rendah dari penyebutnya

b. Faktorkan penyebutnya, menjadi :

- Faktor linier (ax + b) pecahan parsial berbentuk $\frac{A}{ax+b}$
- Faktor linier $(ax + b)^2$ $\rightarrow \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
- Faktor linier $(ax + b)^3$ $\rightarrow \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$
- Faktor linier $(ax^2 + bx + c)$ $\rightarrow \frac{Ax+b}{ax^2+bx+c}$

Contoh :

Bentuk (ax + b) → $\boldsymbol{\frac{A}{ax+b}}$

a. $\int \frac{x+1}{x^2-3x+2}$dx

$= \frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

x + 1 = A (x-2) + B (x-1)

mis : x – 2 = 0 → x = 2 → disubtitusi

2 + 1 = A (2-2) + B (2-1)

3 = B

mis x – 1 = 0 → x = 1 → disubtitusi

1 + 1 = A (1-2) + B(1-1)

2 = -A

A = -2

Jadi

$$\int \frac{x+1}{x^2-3x+2} dx = \int \frac{-2}{x-1} dx + \int \frac{3}{x-2} dx$$

$$= -2 \ln (x-1) + 3 \ln (x-2) + C$$

Bentuk (ax + b) dan (ax + b)²

b. $\int \frac{x^2}{(x+1)(x-1)^2}$

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$x^2 = A (x-1)^2 + B (x+1)(x-1) + C (x+1)$

↗

misal x = 0 → x = 1 disub

$1^2 = A (1-1)^2 + B (1+1) (1-1) + C(x+1)$

1 = 2 C

C = ½

Mis x + 1 = 0 → x = -1 → disub

$(-1)^2 = A (-1-1)^2 + B (-1+1) (-1-1) + C(-1+1)$

1 = 4 A

A = ¼

Pangkat tertinggi x^2

1 = A + B

1 = ¼ + B → B = 1 – ¼ = ¾

Jadi : $\int \frac{x^2}{(x+1)(x-1)^2} dx = \frac{1}{4}\int \frac{1}{x+1} dx + \frac{3}{4}\int \frac{1}{x-1} dx + \frac{1}{2}\int \frac{1}{(x-1)^2} dx$

$$= \frac{1}{4} \ln (x+1) + \frac{3}{4} \ln (x-1) - \frac{1}{2}\frac{1}{x-1} + C$$

↙

$$= \frac{1}{2}\int \frac{1}{(x-1)^2} dx$$

$$= \frac{1}{2}\int (x-1)^{-2}\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{-2+1} (x-1)^{-2+1} + C$$

$$= \frac{1}{2} \cdot \frac{1}{-1} (x-1)^{-1} + C$$

$$= -\frac{1}{2} (x-1)^{-1} + C$$

Bentuk (ax + b)²

b. $\int \frac{x^2+1}{(x+2)^2}$ dx

$\frac{x^2+1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^2}$

$(x^2+1) = A(x+2)^2 + B(x+2) + C$

5 = C

- Koef pangkat tertinggi
  1 = A
- Koef terendah → tidak mengandung variabel
  1 = 4A + 2B + C
  1 = 4.1 + 2B + 5
  1 = 9 + 28
  1-9 = 28
  B = -8/2
  B = -4

Jadi

$\int \frac{x^2+1}{(x+2)^3}\,dx = \int \frac{1}{(x+2)}\,dx - 4\int \frac{1}{(x+2)^2}\,dx + 5\int \frac{1}{(x+2)^3}dx$

$= \ln(x+2) + 4(x+2)^{-1} + 5(x+2)^{-2} + c$

$= \ln(x+2) + 4\frac{1}{(x+2)} - \frac{5}{2} \cdot \frac{1}{(x+2)^2} + c$

$= -4\int \frac{1}{(x+2)^2}\,dx$

$= -4\int (x+2)^{-2}\,dx$

$= -4 \cdot \frac{1}{-2+1}(x+2)^{-2+1} + c$

$= -4 \cdot \frac{1}{-1}(x+2)^{-1} + c$

$= 4(x+2)^{-1} + c$

$= 5\int \frac{1}{(x+2)}\,dx$

$= 5\int (x+2)^{-3}\,dx$

$= 5 \cdot \frac{1}{-3+1}(x+2)^{-3+1} + c$

$= 5 \cdot \frac{1}{-2}(x+2)^{-2} + c$

$= -\frac{5}{2} \cdot (x+2)^{-2} + c$

Bentuk $(ax^2 + bx + c)$

$$\int \frac{x^2}{(x-2)(x^2+1)}\,dx$$

$$\frac{x^2}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+c}{(x^2+1)}$$

$x^2 \quad = A(x^2+1)+(Bx+c)(x-2)$

mis $\quad = x-2 = 0 \rightarrow x=2$ disubtitusi

$(2)^2 \quad = A(2^2+1)+(B.2+c)(2-2)$

$4 \quad = 5A$

$A \quad = \frac{4}{5}$

Pangkat tertinggi = x^2

$1 = A+B$

$1 = \frac{4}{5} + B$

$B = 1-\frac{4}{5}$

$B = \frac{1}{5}$

Koef terendah

$0 = A-2C \rightarrow A = 2C$

$\frac{4}{5} = 2C$

$C = {}^{\frac{4}{5}}\!/_{2}$

$C = {}^{\frac{4}{5}}\!/_{\frac{1}{2}}$

$C = \frac{4}{10}$

$C = \frac{2}{5}$

Jadi

$$\int \frac{x^2}{(x-2)\left(x^2\ \ +1\right)}\ \text{dx} \ = \frac{4}{5}\int \frac{1}{(x-2)}\,\text{dx} + \int \frac{\frac{1}{5}x\frac{2}{5}}{\left(x^2+1\right)}\text{dx}$$

$$= \frac{4}{5}\,ln(x-2) + \frac{1}{5}\int \frac{x}{\left(x^2+1\right)}\,dx + \frac{2}{5}\int \frac{1}{\left(x^2+1\right)}\,dx$$

$$= \frac{4}{5}\,ln(x-2) + \frac{1}{10}\,ln\left(x^2+1\right) + \frac{2}{5}\,tg-1x + C$$

$$= \frac{1}{5}\int \frac{x}{\left(x^2+1\right)}\,dx$$

$$= \frac{1}{5}\int x\left(x^2+1\right)^{-1}\,dx$$

$$= \frac{1}{5}.\frac{x}{1+1}\left(x^2+1\right)^{-1+1} + c = \frac{1}{10}\,ln\left(x^2+1\right)$$

Contoh pemfaktoran

1) $\int \frac{2x+1}{x^3-7x+6}$

$$\frac{2x+1}{x^3\,7x+6} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x+3)}$$

2) $\int \frac{25x^2-20x+2}{x^3-5x^2+4x}$

$$\frac{25x^2-20x+2}{x(x-1)(x-4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-4}$$

3) $\int \frac{\left(x^2-2\right)}{x^3(x+2)^2}$

$$\frac{x^2-2}{x^3(x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x+2)} + \frac{E}{(x+2)^2}$$

4) $\int \frac{1}{x^3+x}$

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

5) $\int \frac{7x^3+20x^2+35x-13}{x^2\left(x^2+4x+13\right)}$

$$\frac{7x^3+20x^2+35x-13}{x^2\left(x^2+4x+13\right)} = \frac{A}{x} + \frac{B}{x} + \frac{cx+d}{x^2+4x+13}$$

→ integral baku no 18

I.7 Integrasi Fungsi fungsi Trigonometri

Rumus dasar dalam Trigonometri

a. $sin^2 x + cos^2 x = 1$

b. $1 + tgx^2 = sec^2 x$

c. $1 + cot\,g^2 x = cos\,e\,c^2 x$

d. $sin^2 x = \frac{1}{2}(1 - cos\,2\,x)$

e. $cos^2 x = \frac{1}{2}(1 + cos\,2\,x)$

f. $sin\,x\,.\,cos\,x = \frac{1}{2} sin\,2\,x$

g. $sin\,x\,.\,cos\,y = \frac{1}{2}(cos(x-y) + sin(x+y))$

h. $cos\,x\,.\,cos\,y = \frac{1}{2}(cos(x-y) + cos(x+y))$

i. $sin\,x\,.\,sin\,y = \frac{1}{2}(cos(x-y) - cos(x+y))$

j. $1 - cos\,x = 2\,sin^2 \frac{1}{2}.x$

k. $1 + cos\,x = 2\,cos^2 \frac{1}{2}x$

l. $1 \pm sin\,x = 1 \pm cos\left(\frac{1}{2}\eta - x\right)$

Contoh:

a. $\int sin^2 x\, dx \;= \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

b. $\int cos^2 x\, dx \;= \frac{1}{2}\int (1+cos\, 2x)\, dx = \frac{x}{2} + \frac{sin\, 2x}{4} + C$

c. $\int sin^3 x\, dx \;= \int sin^2 x \sin x\, dx = \int \left(1-\; cos^2 x\right) \sin x\, dx$

$$= \int \sin x\, dx - \int cos^2 x \sin x\;\; dx$$

$$= -\cos x + \frac{cos^3 x}{3} + C$$

d. $\int cos^4 x\, dx = \int \left(\, cos^2 x\right)^2 dx = \int \frac{(1+\cos 2x)^2}{4} dx$

$$= \int \frac{1 + 2\cos 2x + \; cos^2 2x}{4} dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}.\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}.\cos 4x\right) dx$$

$$= \frac{1}{4}\left\{\frac{3x}{2} + sin\; 2x + \frac{\sin 4x}{8}\right\} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

e. $\int \sin 4x \cos 2x\, dx$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$sehingga : \sin 4x \cos 2x = \frac{1}{2}(2\sin 4x \cos 2x)$

$$= \frac{1}{2}\{\sin(4x+2x) + \sin(4x-2x)\}$$

$$= \frac{1}{2}(\sin 6x + sin\; 2x)$$

$$\int \sin 4x \cos 2x\, dx = \frac{1}{2}\int (\sin 6x + sin\, 2x)\;\; dx = -\frac{\cos 6x}{12} - \frac{\cos 2x}{4} + C$$

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BAB II
DIFERENSIASI

Persamaan diferensial merupakan persamaan yang mengandung derivative (turunan) yang mengandung variable tak bebas satu ataupun lebih terhadap variable bebas satu ataupun lebih dari suatu fungsi. Berikut contoh bentuk persamaan diferensial:

$\frac{dy}{dx}$; $\frac{d^2y}{dx^2}$

variable bebas = x; variable tak bebas= y

II.1 Koefisien Diferensiasi Baku

Diferensial baku adalah suatu persamaan diferensial yang penyelesaiannya langsung menggunakan rumus yang telah ditetapkan. Berikut adalah rumus-rumus penyelesaian diferensial baku.

No	$y = f(x)$	$\frac{dy}{dx}$
1	x^n	$n.x^{n-1}$
2	e^x	e^x
3	e^{kx}	$k.e^{kx}$
4	a^x	$a^x \ln a$
5	$\ln x$	$\frac{1}{x}$
6	${}^a\log x$	$\frac{1}{x \ln a}$
7	$\sin x$	$\cos x$

No	$y = f(x)$	$\frac{dy}{dx}$
8	$\cos x$	$-\sin x$
9	$\tan x$	$sec^2 x$
10	$\cot x$	$-cosec^2 x$
11	$\sec x$	$\sec x \tan x$
12	$cosec\ x$	$-cosec\ x \cot x$
13	$\sinh x$	$\cosh x$
14	$\cosh x$	$\sinh x$

Contoh:

1. x^5	4. a^3	7. $\log 10^x$
2. e^{3x}	5. a^x	8. $\sqrt{x}$
3. 2^x	6. x^{-4}	

Jawab:

1. $x^5 = 5x^4$	5. $a^x = a^x \ln a$
2. $e^{3x} = 3e^{3x}$	6. $x^{-4} = -4x^{-5}$
3. $2^x = 2^x \ln 2$	7. $\log 10x = \frac{1}{x \ln 10}$
4. $a^3 = a^3 \ln a$	8. $\sqrt{x} = x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

II.2 Fungsi Dari Suatu Fungsi

Diferensial fungsi dari suatu fungsi adalah suatu turunan fungsi yang mengandung fungsi yang lain.

Contoh :

1. $y = \cos(5x - 4) \rightarrow \frac{dy}{dx}$?

   misal: $u = 5x - 4$

   $$\frac{du}{dx} = 5$$

   $$y = \cos(5x - 4)$$

   $$\frac{dy}{du} = -\sin(5x - 4)$$

   $$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = -5.\sin(5x - 4) = -5\sin(5x - 4)$$

2. $y = e^{\sin x}$

misal : $u = \sin x \rightarrow \frac{du}{dx} = \cos x$

$$y = e^{\sin x} \rightarrow \frac{dy}{du} = e^{u} = e^{\sin x}$$

$$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = \cos x \,.\, e^{\sin x} = e^{\sin x}.\cos x$$

3. $y = \sin x^{2}$

misal : $u = x^{2}$

$$\frac{du}{dx} = 2x$$

$$y = \sin u$$

$$\frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \,.\, 2\,x$$

$$\frac{dy}{dx} = 2x.\cos x^{2}$$

4. $y = \tan(5x - 4)$

misal : $u = 5x - 4 \rightarrow \frac{du}{dx} = 5$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = 5\sec^{2}(5x - 4)$$

5. $y = ln(3 - 4\,cos\,x)$

misal: $u = 3 - 4\,cos\,x \rightarrow \frac{du}{dx} = 4\,sin\,x$

$$y = ln(3 - 4\,cos\,x) \rightarrow \frac{dy}{du} = {}^{1}\!/\!_{(3 - 4\,cos\,x)}$$

$$\frac{dy}{dx} = \frac{du}{dx}.\frac{d}{du} = 4\,sin\,x.\frac{1}{(3 - 4\,cos\,x)} = \frac{4\,sin\,x}{(3 - 4\,cos\,x)}$$

6. $y = cos\left(x^2\right)$

misal: $u = x^2 \rightarrow {}^{du}\!/\!_{dx} = 2x$

$$y = cos\left(x^2\right) \rightarrow {}^{dy}\!/\!_{du} = -\,sin\,x^2$$

$$\frac{dy}{dx} = \frac{dx}{du}.\frac{du}{dx} = -2x\,sin\,x^2$$

7. $y = e^{3-x}$

misal: $u = 3 - x \rightarrow {}^{du}\!/\!_{dx} = -1$

$$y = e^{3-x} \rightarrow {}^{dy}\!/\!_{du} = e^{3-x}$$

$$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = -1.\,e^{3-x} - e^{3-x}$$

8. $y = (4x - 5)^6$

misal: $u = 4x - 5 \rightarrow {}^{du}\!/\!_{dx} = 4$

$$y = (4x - 5)^6 \rightarrow {}^{dy}\!/_{du} = 6(4x - 5)^5$$

$$\frac{dy}{dx} = \frac{du}{dx} . \frac{dy}{du} = 4.6(4x - 5)^5 = 24(4x - 5)^5$$

II.3 Perkalian Dua Fungsi

Diferensial perkalian dua fungsi merupakan suatu persamaan yang akan diturunkan mempunyai perkalian dua fungsi.

$$y = u.v$$

$$\downarrow$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Contoh:

1. $y = x^3 \sin 3\, x$

   misal: $u = x^3\ {}^{du}\!/_{dx} = 3x^2$

   $$v = \sin 3\, x \rightarrow {}^{dv}\!/_{dx} = 3 \cos 3\, x$$

   $$\frac{dy}{dx} = x^3 . 3 \cos 3\, x + \sin 3\, x . 3x^2$$

   $$\frac{dy}{dx} = 3x^3 \cos 3\, x + \sin 3\, x^2 \sin 3\, x$$

   $$\frac{dy}{dx} = 3x^2 (x \cos 3\, x + \sin 3\, x)$$

2. $y = x^2 \tan x$

   misal: $u = x^2 \rightarrow {}^{du}\!/_{dx} = 2$

$$v = tan\,x \rightarrow {du}/{dx} = sec^2\,x$$

$$\frac{dy}{dx} = x^2\,x + tan\,x\,.\,2x$$

$$\frac{dy}{dx} = x\left(sec^2\,x + 2\,tan\,x\right)$$

3. $y = e^{5x}(3x + 1)$

misal: $u = e^{5x} \rightarrow {du}/{dx} = 5e^{5x}$

$$v = (3x + 1) \rightarrow {du}/{dx} = 3$$

$$\frac{dy}{dx} = e^{5x}.3 + (3x + 1)5e^{5x}$$

$$\frac{dy}{dx} = 3e^{5x} + (3x + 1)5e^{5x}$$

4. $y = x^3\,sin\,5\,x$

misal: $u = x^3 \rightarrow {du}/{dx} = 3x^2$

$$v = sin\,5\,x \rightarrow {du}/{dx} = 5\,cos\,5\,x$$

$$\frac{dy}{dx} x^3\,5\,cos\,5\,x + sin\,5\,x.\,3x^2$$

$$\frac{dy}{dx} = 5x^3\,cos\,5\,x + 3x^2\,sin\,5\,x$$

$$\frac{dy}{dx} = x^2(5x\,cos\,5\,x + 3\,sin\,5\,x)$$

5. $y = x^2 \ln x$

misal: $u = x^2 \rightarrow {}^{du}/_{dx} = 2x$

$v = \ln x \rightarrow {}^{du}/_{dx} = {}^{1}/_{x}$

$$\frac{dy}{dx} = x^2 . \frac{1}{x} + \ln x . 2x$$

$$\frac{dy}{dx} = x + \ln x\, 2x$$

II.4 Pembagian Dua Fungsi

Diferensial pembagian dua fungsi merupakan suatu persamaan yang akan diturunkan mempunyai fungsi pembilang dan penyebut sebagai pembagi.

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Contoh:

1. $y = \frac{\sin 3x}{x+1} \rightarrow u = \sin 3\,x \rightarrow {}^{du}/_{dx} = \frac{1}{x}$

$v = (x + 1) \rightarrow {}^{dv}/_{dx} = 1$

$$\frac{dy}{dx} = \frac{(x + 1). 3 \cos 3\,x - \sin 3\,x. 1}{(x + 1)^2}$$

$$= \frac{3(x + 1) \cos 3\,x - \sin 3\,x}{(x + 1)^2}$$

2. $y = \frac{ln\,x}{e^{2x}} \rightarrow u = ln\,x \rightarrow {du}/{dx} = {1}/{x}$

$v = e^{2x} \rightarrow {du}/{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{e^{2x}.{1}/{x} - ln\,x\,.2e^{2x}}{\left(e^{2x}\right)^2}$$

$$\frac{dy}{dx} = \frac{\frac{e^{2x}}{x} - ln\,x\,2e^{2x}}{e^{4x}}$$

3. $y = \frac{cos\,5x}{e^{3x+2}} \rightarrow u = cos\,5\,x \rightarrow \frac{du}{dx} = -5\,sin\,5\,x$

$v = e^{3x+2} \rightarrow {dv}/{dx} = 3e^{3x+2}$

$$\frac{dy}{dx} = \frac{e^{3x+2}\left(-5\,sin\,5\,x\right) - cos\,5\,x.\,3e^{3x+2}}{\left(e^{3x+2}\right)^2}$$

$$\frac{du}{dx} = \frac{-5e^{3x+2}\,sin\,5\,x - 3e^{3x+2}\,cos\,5\,x}{e^{6x+4}}$$

4. $y = \frac{sin\,3x}{tan\,2v} \rightarrow u = sin\,3\,x = 3\,cos\,3\,x\,{du}/{dx}$

$v = tan\,2\,x = 2\,sec^2\,2\,x\,\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{tan\,2\,x.3\,cos\,3\,x - sin\,3\,x.2\,sec^2\,2\,x}{(tan\,2\,x)^2}$$

$$\frac{dy}{dx} = \frac{3\,tan\,2\,x\,cos\,3\,x - 2\,sin\,3\,x\,sec^2\,2\,x}{(tan\,2\,x)2}$$

II.5 Diferensiasi Logaritmik

Persamaan yang akan dideferensialkan yang kita kenal pada perkalian dan pembagian adalah 2 variabel u.v atau v/u. Jika diketahui ada lebih dari 2 variabel maka dilakukan diferensiasi logaritma.

$$y = \frac{u.w}{v} \Rightarrow$$

$$\ln y = \ln u + \ln w - \ln v$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx} + \frac{1}{w}\frac{dw}{dx} - \frac{1}{v}\frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{w}\frac{dw}{dx} - \frac{1}{v}\frac{dv}{dx}\right]$$

Contoh:

1. $y = \frac{x^2\,sin\,x}{cos\,2x} \rightarrow u = x^2 \rightarrow {du}/{dx} = 2x$

$v = cos\,2\,x = -2\,sin\,2\,x$

$w = sin\,x \rightarrow {dw}/{dx} = cos\,x$

$$\frac{dy}{dx} = \frac{x^2.sin\,x}{cos\,2\,x} = \left(\frac{1}{x^2}.2x + \frac{1}{sin\,x}cos\,x - \frac{1}{cos\,2\,x}(-2\,sin\,2\,x)\right)$$

$$= \left(\frac{2}{x} + \frac{cos\,x}{sin\,x} + 2\frac{sin\,x}{cos\,2\,x}\right)$$

$$= \left(\frac{2}{x} + cos\,x + 2\,tan\,2\,x\right)$$

2. $y = x^4 e^{3x} \tan x \rightarrow u = x^4 \rightarrow du/dx = 4x^3$

$w = e^{3x} \rightarrow dw/dx = 3c^{3x}$

$v = \tan x \rightarrow dv/dx = sec^2 x$

$\frac{dy}{dx} = x^2 . c^{3x} . \tan x = \frac{1}{x^4} . 4x^3 + \frac{1}{e^{3x}} . 3c^{3x} + \frac{1}{\tan x} . sec^2 x$

$= \frac{4}{x} + 3 + \frac{sec^2 x}{\tan x}$

3. $y = \frac{e^{4x}}{x^3 \cos 2x} \rightarrow u = e^{4x} \rightarrow du/dx = 4e^{4x}$

$w = x^3 \rightarrow dw/dx = 3x^2$

$v = \cos 2x \rightarrow dv/dx = 2 \sin 2x$

$\frac{dy}{dx} = \frac{e^{4x}}{x^3 \cos 2x} = \frac{1}{c^{4x} . 4c^{4x}} - \frac{1}{x^3} . 3x^2 - \frac{1}{\cos 2x} . 2 \sin 2x$

$= 4 - \frac{3}{x} - 2 \frac{\sin 2x}{csx}$

$= 4 - \frac{3}{x} - 2 \tan 2x$

4. $y = \frac{(3x+1) \cos 2x}{c^{2x}} \rightarrow u = (3x+1) \rightarrow dv/dx = 3$

$w = \cos 2x \rightarrow dw/dx = -2 \sin 2x$

$v = c^{2x} \rightarrow dv/dx = 2c^{2x}$

$\frac{dy}{dx} = \frac{(3x+1) \cos 2x}{c^{2x}} = \frac{1}{(3x+1)} . 3 + \frac{1}{\cos 2x}(-2 \sin 2x) - \frac{1}{e^{2x}} . 2c^{2x}$

$= \frac{3}{(3x+1)} - 2 \frac{\sin 2x}{\cos 2x} - 2$

$= \frac{3}{(3x+1)} - 2 \tan 2x - 2$

5. $y = x^5 \sin 2x \cos 4x \rightarrow u = x^5 \rightarrow du/dx = 5x^4$

$w = \sin 2x \rightarrow du/dx = 2 \cos 2x$

$v = \cos 4x \rightarrow dv/dx = -4 \sin 4x$

$$\frac{dy}{dx} = x^5 \sin 2x \cos 4x = \frac{1}{x^5} . 5x^4 + \frac{1}{\sin 2x} - \frac{1}{\cos 4x}(-4 \sin 4x)$$

$$= \frac{5}{x} + 2 \cot g\, 2x + 4 \tan 4x$$

II.6 Diferensiasi Fungsi Implisit

Fungsi implisit merupakan perluasan dari fungsi biasa/eksplisit. Misal jika ada suatu persamaan dimana varible x dan y berkaitan sangat erat (tidak terpisah), maka dapat dikatakan terdapat fungsi implisit dalam persamaan tersebut.

$y = x^2 - 4x + 2$	→	$y \rightarrow$ *terdefinisi* x y *disebut fungsi eksplisit dari* x
$xy + \sin y = 2$	→	y *dan* x tidak bisa dipisahkan y disebut fungsi implisit dari x karena (fx) tersirat didalamnya.

Contoh :

1. $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

2. Diketahui persamaan $x^2 + y^2 - 2x - 6y + 5 = 0$, jika titik x = 3, y=2. Tentukan:

   a. Nilai $(\frac{dy}{dx})$

b. Nilai $\left(\frac{d^2y}{dx^2}\right)$

Penyelesaian:

a. $2x + 2y\frac{dy}{dx} - 2 - 6.\frac{dy}{dx} = 0$

$$(2y-6)\frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2-2x}{2y-6}$$

$$\frac{dy}{dx} = \frac{1-x}{y-3}$$

$x = 3, y=2 \rightarrow \frac{dy}{dx} = \frac{1-3}{2-3} = \frac{-2}{-1} = 2$

b. $\frac{d^2y}{dx^2} \rightarrow x = 3, y = 2\ ; \frac{dy}{dx} = 2$

$\frac{d}{dx}\left(\frac{y}{dx}\right) = \frac{d}{dx}\left(\frac{1-x}{y-3}\right)\frac{u}{v} \rightarrow$ $u = 1 - x \Rightarrow du/dx = -1$

$v = y - 3 \Rightarrow dv/dx = dy/dx$

$$= \frac{(y-3)(-1)-(1-x)dy/dx}{(y-3)^2}$$

$$= \frac{(2-3)(-1)-(1-3)^2}{(2-3)^2}$$

$$= \frac{1+4}{1} = 5$$

3. Diketahui persamaan $x^2 + 2xy + 3y^2 = 4$. Tentukan nilai $\frac{dy}{dx}$

$x^2 + 2xy + 3y^2 = 4$

$\rightarrow 2xy \rightarrow$ $u = 2x \rightarrow du/dx = 2$

$v = y \rightarrow dv/dx = dy/dx$

$$\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

$$= 2x\frac{dy}{dx} + 2y$$

$$2x + 2x.\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$$

$$(2x+6)\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x - 24}{2x + 6}$$

$$\frac{dy}{dx} = \frac{-x - 4}{x + 3}$$

II.7 Diferensiasi Persamaan Parametrik

Persamaan parametrik adalah persamaan yang menyertakan variabel x,y pada variabel yang ke tiga secara terpisah.

$$\left.\begin{array}{l} y = \cos 2\,t \\ x = \sin\ \ t \end{array}\right\} \begin{array}{l} t \rightarrow\ parameter \\ x, y \rightarrow pers.\ parametik \end{array}$$

Contoh :

1. Carilah $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ dari y = cos 2t , x = sin t

Penyelesaian:

- y = cos 2t

  $\frac{dy}{dx} = -\ 2\sin 2\,t$

- x = sin t

  $\frac{dx}{dt} = \cos t$

a. $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= -2\sin 2\,t.\frac{1}{\cos t}$

$= \frac{-2\sin 2\,t}{\cos t}$ $\Rightarrow$

$= \frac{-2(2\sin t.\cos t)}{\cos t}$

= -4 sin t

berasal dari rumus
sin x cosx $= \frac{1}{2} 2\sin 2\,x$
sin 2x = 2 sin x cos x
jadi sin 2t = 2 sin t cos t

b. $\frac{d^2y}{dx^2}$ $= \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}(-4\,sin\,t)$

$= \frac{d}{dt}(-4\,sin\,t)\frac{dt}{dx}$

$= -4\,cos\,t.\frac{1}{cos\,t}$

$= -4$

2. $\left.\begin{array}{l} y = 3\,sin\,\theta - sin^3\,\theta \\ x = cos^3\,\theta \end{array}\right\}\frac{dy}{dx}, \frac{d^2y}{dx^2}$?

Penyelesaian:

$y = 3\,sin\,\theta - sin^3\,\theta$

$\frac{dy}{d\theta} = 3\,cos\,\theta - 3\,sin^2\,\theta\,.\,cos\,\theta$

$x = cos^3\,\theta$

$\frac{dy}{d\theta} = 3\,cos\,\theta\,(-\,sin\,\theta)$

$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$ $= \left(3\,cos\,\theta - 3\,sin^2\,\theta\,.\,cos\,\theta\right).\frac{1}{-2\,cos\,\theta\,sin\,\theta}$

$= \frac{3\,cos\,\theta - 3\,sin^2\,\theta.cos\,\theta}{-3\,cos\,\theta\,sin\,\theta}$

$= \frac{3\,cos\,\theta\,\,(1 - sin^2\,\theta)}{-3\,cos^2\,\theta\,sin\,\theta}$

$= \frac{-3\,cos\,\theta\,\,cos^2\,\theta}{3\,cos^2\,\theta\,sin\,\theta}$

$= \frac{-\,cos\,\theta}{sin\,\theta}$

$= -\,cot\,a\,n\theta$

$$\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad = \frac{d}{dx}(-\cot a\, n\theta)$$

$$= \frac{d}{d\theta}(-\cot a\, n\theta)\frac{d\theta}{dx}$$

$$= -\cos e\, c^2\theta . \frac{1}{-\sin\theta 3\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta} . \frac{1}{(\sin\theta)(3\cos^2\theta)}$$

$$= \frac{1}{3\sin^3\theta\cos^2\theta}$$

BAB III
FUNGSI GAMMA DAN BETA

Fungsi Gamma dan Beta merupakan fungsi-fungsi istimewa yang sering muncul dalam pemecahan persamaan differensial, proses fisika, perpindahan panas, gesekan sumber bunyi, rambatan gelombang, potensial gaya, persamaan gelombang, mekanika kuantum, dan lainnya. Fungsi Gamma dan Beta merupakan fungsi dalam bentuk pernyataan integral dan mudah untuk dipelajari. Kedua fungsi ini biasanya dibahas secara rinci dalam fungsi bilangan kompleks (di sini hanya dibahas secara definisi dan sifat-sifat sederhana yang dimiliki fungsi tersebut).

Fungsi Gamma

Didefinisikan dalam bentuk :

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx = \lim_{b\to\infty} \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

→ Konvergen untuk $n > 0$

Hasil integral menyatakan bahwa $\Gamma(n) = (n-1)!$

$\Gamma(n)$ disebut juga fungsi factorial atau perkalian berlanjut dengan $n = 1, 2, 3, \ldots$

Contoh :

$\Gamma(1) = ?$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx = \lim_{b\to\infty} \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x}\, dx = \lim_{b\to\infty} \int_0^{\infty} x^{1-1} e^{-x}\, dx$$

$$= \lim_{b\to\infty} \int_0^{\infty} e^{-x}\ dx$$

$$= \lim_{b\to\infty} \left[-e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty} \left[-e^{-b} + e^{-0}\right] = 1$$

Jadi $\mathbf{\Gamma(1) = 1}$

Rumus rekursi dari fungsi Gamma :

$$\Gamma(n+1) = n\Gamma(n)$$

Dimana $\Gamma(1) = 1$

Contoh :

1. $\Gamma(2) = (1+1) = 1\Gamma(1) = 1$
2. $\Gamma(3) = (2+1) = 2\Gamma(1) = 2$
3. $\Gamma(3/2) = (1/2+1) = 1/2\Gamma(1) = 1/2$

Bila **n** bilangan bulat positif :

$$\Gamma(n+1) = n!$$

Dimana $\Gamma(1) = 1$

Contoh :

1. $\Gamma(2) = (1+1) = 1! = 1$
2. $\Gamma(3) = (2+1) = 2! = 2$
3. $\Gamma(4) = (3+1) = 3! = 6$

Bila **n** bilangan pecahan positif :

$$\Gamma(n) = (n-1).(n-2).\ldots\ \alpha\Gamma(\alpha)$$

Contoh :

1. $\Gamma(3/2) = (1/2)\,\Gamma(1/2)$
2. $\Gamma(7/3) = (4/3)\,(1/3)\,\Gamma(1/3)$
3. $\Gamma(7/2) = (5/2)\,(3/2)\,(1/2)\,\Gamma(1/2)$

Bila **n** bilangan pecahan negatif :

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

atau

$$\Gamma(n) = \frac{\Gamma(n+m)}{n(n-1)\ldots}$$

Contoh :

1. $\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma(-1/2)}{-3/2} = \frac{\Gamma(-1/2+1)}{(-3/2)(-1/2)} = \frac{\Gamma(1/2)}{(-3/2)(-1/2)} = \frac{\Gamma(1/2)}{(3/4)} = \frac{4\,\Gamma(1/2)}{3}$

2. $\Gamma(-5/2) = \frac{\Gamma(-5/2+1)}{-5/2} = \frac{\Gamma(-3/2)}{-5/2} = \frac{\Gamma(-3/2+1)}{(-5/2)(-3/2)} = \frac{\Gamma(-1/2)}{(-5/2)(-3/2)} = \frac{\Gamma(-1/2+1)}{(-5/2)(-3/2)(-1/2)} =$
   $\frac{\Gamma(1/2)}{(-15/8)} = -\frac{8\,\Gamma(1/2)}{15}$

Beberapa hubungan dalam fungsi Gamma :

- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(n) = (n-1)\,!$
- $\Gamma(n) = \frac{\Gamma(n+1)}{n}$
- $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Soal Latihan :

Hitunglah $\int_0^\infty x^6 e^{-2x}\,dx$

Jawab :

Misal : $2x = y \quad \rightarrow \quad x = \frac{1}{2}y \quad \rightarrow \quad dx = \frac{1}{2}dy$

Jika $x = 0$, maka $y = 0$

Jika $x = \infty$, maka $y = \infty$

$$\int_0^\infty x^6 e^{-2x}\,dx = \int_0^\infty \left(\frac{1}{2}y\right)^6 e^{-y}\,\frac{1}{2}dy = \int_0^\infty \left(\frac{1}{2}\right)^7 y^6\,e^{-y}\,dy$$

$$= \left(\frac{1}{2}\right)^7 \int_0^\infty y^6\,e^{-y}\,dy = \left(\frac{1}{2}\right)^7 \int_0^\infty y^{7-1}\,e^{-y}\,dy$$

$$= \left(\frac{1}{2}\right)^7 \Gamma(7) = \frac{6!}{2^7} = \frac{45}{8}$$

Fungsi Beta

Didefinisikan dalam bentuk :

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

→ Konvergen untuk $m > 0$ dan $n > 0$

Sifat :

$$\beta(m,n) = \beta(n,m)$$

Bukti :

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-y)^{m-1}\,(y)^{n-1}\,dx$$

$$= \int_0^1 y^{n-1}\,(1-y)^{m-1}\,dx$$

$$= \beta(n,m) \qquad \rightarrow \boldsymbol{terbukti}$$

Hubungan fungsi Beta dan fungsi Gamma :

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Contoh :

1. $\beta(3{,}5) = \frac{\Gamma(3)\,\Gamma(5)}{\Gamma(3+5)} = \frac{\Gamma(3)\,\Gamma(5)}{\Gamma(8)} = \frac{2\,!\;4\,!}{7\,!} = \frac{48}{5040} = \frac{1}{105}$

2. $\int_0^1 x^4\,(1-x)^3\,dx = \int_0^1 x^{5-1}\,(1-x)^{4-1}\,dx$

$$= \beta(5{,}4) = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(5+4)} = \frac{4!\,3!}{8!} = \frac{144}{40320} = \frac{1}{280}$$

Soal-soal dan Penyelesaian

1. Buktikan $\Gamma(1) = 1$

   Penyelesaian :

   Diket: $\Gamma(1) = 1$

   $\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx =$
   $\lim_{b\to\infty} \int_0^b x^{1-1}e^{-x}dx$

   $= \lim_{b\to\infty} \int_0^b e^{-x}dx$

   $= \lim_{b\to\infty}[-e^{-x}]_0^b$

   $= \lim_{b\to\infty}[-e^{-b} + e^0] = 1$

2. Hitunglah $\Gamma(7) = \ldots.$

   Penyelesaian:

   Diket : $\Gamma(7)$

   $\Gamma(n+1) = n!$

   $\Gamma(7) = \Gamma(6+1) = 6! = 720$

3. Hitunglah $\Gamma\left(-\frac{3}{2}\right)$

   Penyelesaian :

   Diket : $\Gamma\left(-\frac{3}{2}\right)$

   $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

   $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}}$

   $= \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(-\frac{1}{2}+1)}{(-\frac{3}{2})(-\frac{1}{2})} = \frac{\Gamma(\frac{1}{2})}{\frac{3}{4}}$

Penyelesaian :

Misal $y = 2x \to dy = 2dx$

$dx = \frac{1}{2}dy$

Jika $x = 0 \quad y = 0$

$x = \infty y = \infty$

$\int_0^\infty x^6 e^{-2x}dx = \int_0^\infty (\frac{1}{2}y)^6 e^{-y}\frac{1}{2}dy$

$= (\frac{1}{2})^7 \int_0^\infty y^6 e^{-y}dy$

$= (\frac{1}{2})^7 \int_0^\infty y^{7-1} e^{-y}dy$

$= \left(\frac{1}{2}\right)^7 \Gamma(7) = (\frac{1}{2})^7 \cdot 6! = \frac{6!}{2^7} = \frac{720}{128}$

6. Hitung $I = \int_0^\infty \frac{x^4}{(1+x)^7}dx$

   Penyelesaian :

   Diket : $\beta(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}$

   $p - 1 = 4$

   $p = 5$

   $p + q = 7$

   $5 + q = 7$

   $q = 2$

$$\beta(p,q) = \frac{\Gamma(5)\Gamma(2)}{\Gamma(5+2)}$$

4. Hitunglah $\frac{\Gamma(5/2)}{\Gamma(1/2)}$

Penyelesaian :

Diket : $\frac{\Gamma(5/2)}{\Gamma(1/2)}$

$$\frac{\Gamma(5/2)}{\Gamma(1/2)} = \frac{(3/2)(1/2)\Gamma(1/2)}{\Gamma(1/2)} = \frac{3}{4}$$

5. Hitung $\int_0^{\infty} x^6\, e^{-2x} \mathrm{dx}$

$p = 6$

$q - 1 = 4$

$q = 5$

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\beta(6,5) = \frac{\Gamma(6)\Gamma(5)}{\Gamma(6+5)}$$

$$= \frac{5!4!}{\Gamma(11)} = \frac{5!4!}{10!}$$

$$= \frac{5!(4\cdot3\cdot2\cdot1)}{10\cdot9\cdot8\cdot7\cdot6\cdot5!}$$

$$= \frac{24}{30240} = \frac{1}{1260}$$

8. Hitunglah $\frac{\Gamma(3)\Gamma(2,5)}{\Gamma(4,5)}$

Penyelesaian :

$$\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(4.5)} = \frac{2!\cdot(1.5)(0.5)\Gamma(0.5)}{(3.5)(2.5)(1.5)(0.5)\Gamma(0.5)}$$

$$= \frac{2}{(3.5)(2.5)}$$

9. Hitunglah $\frac{6\Gamma(7/3)}{5\Gamma(2/3)}$

Penyelesaian:

$$\frac{6\Gamma(7/3)}{5\Gamma(2/3)} = \frac{6\cdot(4/3)(1/3)\Gamma(1/3)}{5\cdot\Gamma(2/3)}$$

$$= \frac{4!1!}{\Gamma(7)} = \frac{4!}{6!}$$

$$= \frac{4!}{6\cdot5\cdot4!} = \frac{1}{30}$$

7. Hitung $\mathrm{I} = \int_0^1 x^5(1-x)^4\, dx$

Penyelesaian :

$$\beta(p,q) = \int_0^1 x^5(1-x)^4 dx$$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$$

$p - 1 = 5$

$$= \Gamma(\tfrac{5}{3})$$

$$= \tfrac{2}{3}\Gamma(\tfrac{2}{3})$$

11. Hitunglah $\int_0^1 x^9(1-x)^7\, dx$

Penyelesain :

Bentuk Umum $\beta(p,q) = \int_0^1 x^9(1-x)^7 dx$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$$

$p - 1 = 9$

$p = 10$

$q - 1 = 7$

$q = 8$

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\beta(10,7) = \frac{\Gamma(10)\Gamma(8)}{\Gamma(10+8)}$$

$$= \frac{9!7!}{\Gamma(18)} = \frac{9!7!}{17!}$$

$$= \frac{9!(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)}{17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9!}$$

$$= \frac{5040}{980179200} = \frac{1}{194480}$$

10. Hitunglah $\int_0^\infty x^{\frac{2}{3}}\, e^{-x}$ dx

Penyelesaian :

Bentuk umum: $\int_0^\infty x^{n-1}e^{-x}dx =$ Γ(n)

n-1= $\frac{2}{3}$

n= $\frac{2}{3} + \frac{3}{3}$

n= $\frac{5}{3}$

$\int_0^\infty x^{\frac{2}{3}}\, e^{-x}$ dx

= Γ(n) dengan n = $\frac{5}{3}$

12. Hitunglah $\frac{\frac{2}{3}\Gamma(^5/_2)}{\frac{3}{2}\Gamma(^3/_2)}$

Penyelesaian:

$$\frac{\frac{2}{3}\Gamma(^5/_2)}{\frac{3}{2}\Gamma(^3/_2)} = \frac{\frac{2}{3}\cdot(^3/_2)(^1/_2)\Gamma(^1/_2)}{\frac{3}{2}\cdot(^1/_2)\Gamma(^1/_2)}$$

$$= \frac{\frac{2}{3}\cdot(^3/_2)(^1/_2)}{\frac{3}{2}\cdot(^1/_2)}$$

$$= \frac{^6/_{12}}{^3/_4} = \frac{2}{3}$$

BAB IV
DERET

Barisan adalah himpunan besaran U_1, U_2 disusun dalam urutan tertentu dan masing-masing suku dibentuk menurut pola tertentu pula yaitu Ur = f(r).

Contoh : 1,3,5,7

2,6,18,54…

Deret dibentuk oleh jumlah suku-suku besaran.

contoh : 1,3,7,5,7 ...barisan

1+3+5+7 ...deret

A. Deret hitung (DH)

a + (a+d) + (a+ 2d) + a + 3d) + ...

a = suku pertama

d = beda

(i) **suku(U) ke – n = a + (n-1)d**

(ii) jumlah n buah suku yang pertama

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

Contoh :

Jika suku ke-7 suatu deret hitung (DH) adalah 22, suku ke 12 adalah 37 tentukan deret tersebut !

Jawab :

$$U_n = a + (n-1)\,d$$

$$U_7 = a + 6\,d = 22 \qquad d = 3 \Rightarrow a + 6\,d = 22$$

$$U_{12} = \underline{a + 11\,d = 37} \qquad a + 6(3) = 22$$

$$-5\,d = -15 \qquad a + 18 = 22$$

$$d = 3 \qquad a = 4$$

maka deretnya : 4 + 7 + 10 + 13 + 16 + …

Latihan

1. Suku ke 6 DH = -5

   Suku ke 10 DH = -21

   Tentukan jumlah 30 suku deret pertama.

   Jawab :

   $U_6 = a + 5d = -5$ | $d = -4 \Rightarrow a + 5d = -5$

   $U_{10} = \underline{a + 9d = -21}$ | $a + 5(-4) = -5$

   $-4d = 16$ | $a = 15$

   $d = -4$

   $S_n = \frac{n}{2}(2a + (n-1)d)$

   $S_{30} = \frac{30}{2}(2.15 + (30-1) - 4)$

   $S_{30} = 15\ (30\text{-}116)$

   $= 15\ (-86)$

   $\mathbf{S_{30} = -1290}$

   Maka jumlah 30 suku deret pertama yaitu -1290.

2. Pada tahun pertama sebuah butik memproduksi 400 stel jas. Setiap tahun rata-rata produksinya bertambah 25 stel jas. Berapakah banyaknya stel jas yang diproduksi pada tahun ke-5 dan berapa total jas yang diproduksi pada tahun ke-5?

   Penyelesaian:

   Banyaknya produksi tahun ke 1, 2, 3, dan seterusnya membentuk suatu barisan yaitu: 400, 425, 540, ….

   A = 400 dan d = 25, sehingga

   $U_5 = a + 4d$

   $= 400 + 4(25)$

   $= 400 + 100$

   $\mathbf{U_5 = 500}$

   Jadi banyaknya produksi pada tahun ke-5 adalah 500 stel jas.

   Total produksi pada tahun ke-5 adalah

   $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

   $S_n = 5/2\ (2(400)+(5\text{-}1)25)$

   $\mathbf{S_n = 2250}$

B. Deret Ukur (DU)

$a + ar + ar^2 + a^3 + ...$

(i) **suku ke – n = ar^{n-1}**

(ii) jumlah suku yang pertama

$$S_n = \frac{a(1-r)^n}{1-r}$$

Contoh :

$8 + 4 + 2 + 1 + ½ + ...$dst

tentukan jumlah 8 buku pertama

Diketahui : $r = \frac{4}{8} = \frac{1}{2}$

$a = 8$

Jawab :

$$S_8 = \frac{8\left(1-\frac{1}{2}\right)^8}{1-{}^{1}\!/_{2}}$$

$$= 15\frac{15}{16}$$

Latihan !

1. Jika suku ke 5 DU = 162
   Suku ke 8 DU = 4374
   Tentukan deret tersebut !
   Jawab :
   $U_5 = ar^4 = 162$
   $U_8 = ar^7 = 4374$
   $$\frac{ar^7}{ar^4} = \frac{4374}{162}$$
   $r^3 = 27$
   $r = 3 \Rightarrow ar^4 = 162$
   $a(3)^4 = 162$

a $= 2$

maka : a + ar + ar² + ar³ + ...

2 + 6 + 18 + 54 + 162 + ...

2. Diketahui suatu deret 2 + 6 + 18 + 54 + ...

   Tentukan suku ke-10 dan jumlah 10 suku pertama!

   Penyelesaian:

   a = 2

   r = 6/2 = 3

   suku ke-10 = $ar^9 = 2(3)^9$

   jumlah 10 suku pertama :

   $$\frac{a\left(1-r^{10}\right)}{1-r} = \frac{2\left(1-(3)^{10}\right)}{1-3} = \frac{2(1-59049)}{-2} = 59048$$

C. Deret Pangkat Bilangan Asli

Deret $1 + 2 + 3 + 4 \ldots \sum_1^n r$

Deret hitung dengan a = 1

d = 1

$$\sum_1^n r = 1 + 2 + 3 + 4 + 5 \ldots .+ n$$

$$= \frac{n}{2}\left(2a + (n-1)d\right) = \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}$$

jadi kesimpulannya Deret pangkat bilangan asli

$$\sum_1^n r = \frac{n(n+1)}{2}$$

$$\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_1^n r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Contoh :

1) Hitung jumlah deret $\sum_{n=1}^{5} n(3+2n)$

Jawab :

$S_5 \quad = \sum_{1}^{5} n(3+2n)$

$= \sum_{1}^{5} (3n + 2n^2)$

$= \sum_{1}^{5} 3n + \sum_{1}^{5} 2n^2$

$= 3\sum_{1}^{5} n + 2\sum_{1}^{5} n^2$

$= 3\left(\frac{n(n+1)}{2}\right) + 2\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= 3\left(\frac{5.6}{2}\right) + 2\left(\frac{5(6)(11)}{6}\right)$

$= 45 + 110$

$S_5 \quad = 155$

2) $\sum_{n=1}^{4} 2n + n^3$

$S_4 \quad = \sum_{n=1}^{4} 2n + n^3$

$= 2\sum_{n=1}^{4} n + \sum_{n=1}^{4} n^3$

$= 2\left(\frac{n(n+1)}{2}\right) + \left(\frac{n(n+1)}{2}\right)^2$

$= 2\left(\frac{4.5}{2}\right) + \left(\frac{4(5)}{2}\right)^2$

$= 20 + 100$

$S_4 \quad = 120$

D. Deret Tak Berhingga

Contoh : $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.$

deret ukur : a = 1, r = ½

$$S_n = \frac{a(1-r)^n}{1-r}$$

$$= \frac{1(1-{}^1/_2)^n}{1-{}^1/_2} = 2(1 - \frac{1}{2})$$

Jika n sangat besar, 2^n sangat besar, maka $\frac{1}{2^n}$ sangat kecil.

Jika $n \rightarrow \sim, \frac{1}{2^n} \rightarrow 0$

Jumlah semua suku dalam deret tak behringga diberikan oleh:

S~ = hanya limit batas

$S_n = \underset{n\rightarrow\sim}{Lt}\,\{Sn\} = 2(1-0) = 2$

Ini berarti kita membuat deret sedekat mungkin dengan 2 dengan mengambil banyak suku yang cukup banyak.

Deret hitung : 1 + 3 + 5 + 7 ....
a = 1, d = 2

$Sn = n^2$

Jika n besar → Sn besar pula

Jika n → ~ → Sn→~

Pada deret hitung → jika mencari jumlah tak terhingga akan diperoleh

+ ~

- ~.

Adakalanya dapat menghitung jumlah tak terhingga suku deret ukur:

$Sn = \frac{a(1-r^n)}{1-r}$ jika $|r| < 1$ maka untuk $r^n \to 0$ untuk $n \to \sim$

Sehingga $S\sim = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

$$\boxed{S\sim = \frac{a}{1-r}}$$

Contoh :

1. $20 + 4 + 0{,}8 + 0{,}16 + 0{,}032 + ...$

   $S\sim = ...?$

   Jawab :

   $a = 20$

   $r = \frac{0{,}8}{4} = 0{,}2 = \frac{1}{5} \to |r| < 1$

   $S\sim = \frac{a}{1-r} = \frac{20}{1-{}^{1}/_{5}} = \frac{5}{4}.\,20$

   $S\sim = 25$

2. Suatu deret tak hingga jumlahnya 20 dan suku pertamanya 10. Hitunglah jumlah 6 suku pertamanya.

   Penyelesaian:

   $S\sim = \frac{a}{1-r}$

   $20 = \frac{10}{1-r}$

   $20(1-r) = 10$

   $20 - 20r = 10$

   $20r = 10$

   $r = \frac{10}{20} = \frac{1}{2}$

   $S_6 = \frac{a(1-r^n)}{1-r}$

$S_6 \qquad = \dfrac{10(1-\left(\frac{1}{2}\right)^6)}{1-\frac{1}{2}}$

$S_6 \qquad = \dfrac{\mathbf{315}}{\mathbf{16}} = 19\,\dfrac{11}{16}$

RANGKUMAN

1. Deret Ukur : a + ar + ar^2 + ar^3+ + ...

   $U_n = ar^{\,n-1}$

   $S_n = \frac{a(1-r)^n}{1-r}$

2. Deret Hitung : a + (a+d) + (a + 2d) + a + 3d)

   $U_n = a + (n-1)d$

   $S_n = \frac{n(n+1)}{2}$

3. Deret pangkat bilangan asli

   $\sum_1^n r = \frac{n(n+1)}{2}$

   $\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$

   $\sum_1^n r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

4. Deret tak terhingga : Sn = U1 + U2 + ... Un

   jika $\underset{n\to\sim}{Lt}\ S_n$ memberikan harga tertentu ⇒ konvergen

   $\underset{n\to\sim}{Lt}\ S_n$ tidak memberikan harga tertentu ⇒ divergen

5. Kaidah Uji Kokeonvergenan

   a. Jika $\underset{n\to\sim}{Lt}\ U_n = 0$, deret mungkin konvergen

   $\underset{n\to\sim}{Lt}\ U_n \neq 0$, deret pasti divergen

b. Uji perbandingan – deret perbandingan yang penting

$$\frac{1}{p^1}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots\ldots+\frac{1}{p^n}$$

Jik P > 1 deret konvergen

P < deret divergen

c. Uji pembagian D'alembert untuk deret-deret bersuku positif

Jika $\underset{n\to\sim}{Lt}\ \frac{U_n+1}{U_n} < 1$ deret konvergen

$\underset{n\to\sim}{Lt}\ \frac{U_n+1}{U_n} > 1$ deret divergen

$\underset{n\to\sim}{Lt}\ \frac{U_n+1}{U_n} = 1$ Tidak ada kesimpulan

d. Deret pada umumnya

1) Jika $\sum|U_n|$ konvergen, maka $\sum U_n$, konvergen mutlak
2) Jika $\sum|U_n|$ divergen, tetapi $\sum U_n$, konvergen maka $\sum U_n$,, konvergen bersyarat

BAB V
LIMIT FUNGSI

Limit digunakan didalam kalkulus untuk mencari turunan dan kekontinyuan. Limit fungsi adalah salah satu konsep mendasar dalam kalkulus dan analisis, tentang sifat suatu fungsi mendekati titik masukan tertentu.

$$\lim_{x \to a} f(x) = L$$

Definisi : "Limit *f(x)* Ketika **x mendekati a** sama dengan *L,* **tetapi x ≠ a**".

Kalimat "**tetapi x ≠ a**" menunjukkan bahwa dalam menentukan limit f(x) Ketika x mendekati a, tidak pernah dianggap x = a. bahkan tidak harus terdefinisi pada x = a. Hal yang diperlukan adalah bagaimana f didefinisikan didekat a.

Contoh :

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Pembahasan :

Agar fungsi dapat terdefnisi untuk sebuah pecahan : $\frac{a}{b}$, maka b ≠ 0, sehingga x ≠1.

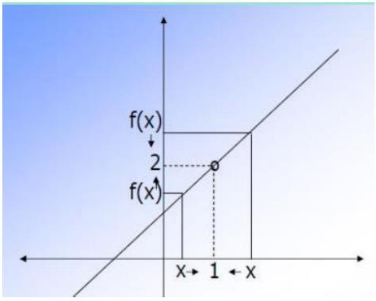
Karena di titik tersebut f(x) berbentuk $\frac{0}{0}$. Tetapi masih dapat dihitung beberapa nilai f(x) jika x mendekati 1.

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01
f(x)	1.9	1.99	1.999	1.9999	∞	2.0001	2.001	2.01

Sehingga dari tabel dan grafik disamping terlihat bahwa f(x) mendekati 2 jika x mendekati 1 yang secara matematis dapat dituliskan sebagai berikut:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

Dibaca: Limit dari $\frac{x^2-1}{x-1}$ untuk nilai x mendekati 1 adalah 2.



TEOREMA LIMIT

1. Limit sebuah konstanta nilainya adalah tetap.

   $$\lim_{x\to c} k = k$$

   Contoh :

   $$\lim_{x\to c} 4 = 4$$

2. $$\lim_{x\to c} x = c$$

   Contoh :

   $$\lim_{x\to 3} 2x = 2\,x\,3 = 6$$

3. $$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)]$$

   Contoh :

   $$\lim_{x\to 5}[x+9] = \lim_{x\to 5} x + \lim_{x\to 5} 9 = 5 + 9 = 14$$

4. $$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

   Contoh :

   $$\lim_{x\to -2}[x - 5] = \lim_{x\to -2} x - \lim_{x\to -2} 5 = -2 - 5 = -7$$

5. $\mathbf{\lim_{x \to c}[f(x).\,g(x)] = \lim_{x \to c} f(x).\lim_{x \to c} g(x)}$

Contoh :

$\lim_{x \to 5}[(8 - x).(x + 4)] = \lim_{x \to 5} f(8 - x).\lim_{x \to 5}(x + 4)$

$= (8 - 5)\ (5 + 4) = 27$

6. $\mathbf{\lim_{x \to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}}$

Contoh :

$\lim_{x \to 4}\left[\frac{x}{3 - x}\right] = \frac{\lim_{x \to 4} x}{\lim_{x \to 4}(3 - x)} = \frac{4}{3 - 4} = -4$

7. $\mathbf{\lim_{x \to c} af(x) = a\lim_{x \to c} f(x)}$

Contoh :

$\lim_{x \to e} 9x = 9\lim_{x \to e} x = 9e$

8. $\mathbf{\lim_{x \to c}[f(x)]^n = \left[\lim_{x \to c}[f(x)]^n\right]}$

Contoh :

$\lim_{x \to 2}(x - 3)]^7 = \left[\lim_{x \to 2}[(x - 3)]^7\right] = (-1)^7 = -1$

9. $\mathbf{\lim_{x \to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}}$

Asalkan $\lim_{x \to c} f(x) > 0$ untuk n bilangan genap.

LIMIT PADA POLINOMIAL

Jika p(x) dan q(x) adalah polynomial, maka:

- $\lim_{x \to c} (x) = p(c)$
- $\lim_{x \to c} \frac{p(x)}{q(c)} = \frac{p(x)}{q(c)}$, untuk $q \neq 0$

Penyelesaian limit:

a. Substitusi Langsung

- $\lim_{x \to 4} \frac{2x - 2}{x + 2} = \frac{2(4) - 2}{4 + 2} = \frac{6}{6} = 1$
- $\lim_{x \to 3} \frac{x^3 - 2}{3x - 4} = \frac{(3)^3 - 2}{3(3) - 4} = \frac{25}{5} = 5$

b. Faktorisasi

- $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2} = \lim_{x \to 2} \frac{(2x + 1)(x - 2)}{(x - 2)} = \lim_{x \to 2} (2x + 1)$

$$= 2(2) + 1 = 5$$

- $\lim_{x \to 1} \frac{x^2 + 9x - 10}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 10)}{(x - 1)} = \lim_{x \to 1} (x + 10)$

$$= (1) + 10 = 11$$

c. Perkalian Sekawan

- $\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} \text{ x } \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$

$$\lim_{x \to 9} \frac{(x - 9)(\sqrt{x} + 3)}{(x - 9)} = \sqrt{9} + 3 = 6$$

- $\lim_{x \to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4} = \lim_{x \to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4} \; x \; \frac{\sqrt{x^2 + 7} + 4}{\sqrt{x^2 + 7} + 4}$

$$\lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{x^2+7-16} = \lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{(x^2-9)}$$

$$\lim_{x\to 3} \sqrt{x^2+7}+4 = \sqrt{(3)^2+7}+4 = 16$$

d. Diferensiasi (L'Hospital Rule)

$$\boldsymbol{lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(x)} = \frac{0}{0},} \text{ maka berlaku}$$

$$\boldsymbol{lim_{x\to c} \frac{p(x)}{q(x)} = \lim_{x\to c} \frac{p'(x)}{q'(x)} = \frac{p'(c)}{q'(c)}}$$

Contoh :

1. $lim_{x\to 3} 5x^2 = 5(3)^2 = 45$
2. $lim_{x\to 3}(5x^2 - 20) = 5(3)^2 - 20 = 25$
3. $lim_{x\to 3} \frac{\sqrt{5x^2-20}}{x} = lim_{x\to 3} \frac{\sqrt{5x^2-20}}{x} x \frac{x}{x} = \frac{\sqrt{5(3)^2-20}}{3} = \frac{5}{3}$
4. $lim_{x\to 2} \frac{x^2+3x-10}{x^2+x-6} = lim_{x\to 2} \frac{(x+5)(x-2)}{(x+3)(x-2)} = lim_{x\to 2} \frac{(x+5)}{(x+3)} = \frac{(2+5)}{(2+3)} = \frac{7}{5}$
5. $lim_{x\to 3} \frac{x^2+3x-18}{x^2-3x} = lim_{x\to 3} \frac{(x+6)(x-3)}{x(x-3)} = lim_{x\to 3} \frac{(x+6)}{x} = \frac{(3+6)}{3} = 3$
6. $lim_{x\to 2} \frac{2x^2-3x+2}{x-2} = lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)} = lim_{x\to 2} \frac{(x-1)}{x} = \frac{(2-1)}{2} = \frac{1}{2}$
7. $lim_{x\to -2} \frac{x^2-x-6}{x^2+3x+2} = lim_{x\to -2} \frac{(x+2)(x-3)}{(x+2)(x+1)} = l \quad m_{x\to -2} \frac{(x-3)}{(x+1)} = \frac{(-2-3)}{(-2+1)} = 5$
8. $lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} x \frac{x+4}{x+4} = lim_{x\to 4} \frac{(\sqrt{x}-2)(x+4)}{(x^2-16)} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(x+4)}{(x-4)(x+4)}$

$$\lim_{x\to 4} \frac{(\sqrt{x}-2)}{(x-4)} = \frac{(\sqrt{4}-2)}{(4-4)} = 0$$

BAB VI
DERET PANGKAT

- Deret pangkat untuk x=0

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

- Deret pangkat untuk x=a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + ..$$

Dimana a merupakan **pusat** dan c_0, c_1, c_2 , ..., c_n merupakan **konstanta,** sedangkan x merupakan variabel.

1. Deret Binomial

   Teorema Binomial (n = bilangan positif)

   $$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

   ✓ $$(1+x)^n = 1 + xn + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

   $$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

   ✓ $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

   ✓ $$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$$

   ✓ $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

   ✓ $$e^{-x} = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Harga Limit → bentuk-bentuk tak tentu(memasukan deret)

a. $\lim_{x \to 0} \left(\frac{x^2+5x-14}{x^2-5x+8} \right) = \frac{-14}{8} = -\frac{7}{2}$

$$\lim_{x \to 0} \left(\frac{2x^3 + 5x}{2x^2 - 5x} \right) = \lim_{x \to 0} \left(\frac{6x^2 + 5}{4x - 5} \right) = \frac{5}{-5} = -1$$

b. $\lim_{x \to 0} \left\{ \frac{\tan x - x}{x^3} \right\} = \lim_{x \to 0} \left\{ \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \right) - x}{x^3} \right\}$

$$= \lim_{x \to 0} \left\{ \frac{\frac{x^3}{3} + \frac{2x^5}{15} + \ldots}{x^3} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{1}{3} + \frac{2x^2}{15} + \ldots \right\}$$

$$= \frac{1}{3}$$

c. $\lim_{x \to 0} \left\{ \frac{\sin^2 x}{x^2} \right\} = \lim_{x \to 0} \left\{ x^2 - \frac{x^4}{3} + \frac{2x^6}{45} \right\}$

$$= \lim_{x \to 0} \left\{ 1 - \frac{x^2}{3} + \frac{2x^4}{45} \right\}$$

$$= 1$$

d. $\lim_{x \to 0} \left\{ \frac{\sin x - x}{x^3} \right\} = \lim_{x \to 0} \left\{ \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \right) - x \right\}$

$$= \lim_{x \to 0} \left\{ \frac{1}{3} + \frac{x^2}{5!} + \frac{x^4}{7!} \right\}$$

$$= \frac{1}{3!}$$

$$= \frac{1}{3x2x1} = \frac{1}{6}$$

2. Kaidah L Hopital

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to a}\left\{\frac{f^1(x)}{g^1(x)}\right\}$$

a. $\lim_{x\to 1}\left\{\frac{x^3+x^2-x-1}{x^2+2x-3}\right\} = \left\{\frac{1+1-1-1}{1+2-3}\right\} = \frac{0}{0}(langsung)$

$$\lim_{x\to 1}\left\{\frac{x^3+x^2-x-1}{x^2+2x-3}\right\} = \lim_{x\to 1}\left\{\frac{3x^2+2x-1}{}\right\}$$

$$\left\{\frac{3+2-1}{2+2}\right\} = \frac{4}{4} = 1$$

b. $\lim_{x\to 0}\left\{\frac{\cosh x - e^x}{x}\right\} = \frac{0}{0}(langsung)$

$$\lim_{x\to 0}\left\{\frac{\cosh x - e^x}{x}\right\} = \lim_{x\to 0}\left\{\frac{\sinh x - e^x}{1}\right\}$$

$$= \frac{-1}{1} = -1$$

c. $\lim_{x\to 0}\left\{\frac{x^2-\sin 3x}{x^2+4x}\right\} = \lim_{x\to 0}\left\{\frac{2x-3\cos 3x}{2x+4}\right\}$

$$= \frac{0-3}{4} = \frac{-3}{4}$$

d. $\lim_{x\to 0}\left\{\frac{\sinh x - \sin x}{x^3}\right\} = \frac{0}{0}$

$$\lim_{x\to 0}\left\{\frac{\cosh x - \cos x}{3x^2}\right\} = \frac{0}{0}$$

$$\lim_{x\to 0}\left\{\frac{\sin n\, x - \sin x}{6x}\right\} = \frac{0}{0}$$

$$\lim_{x\to0}\left\{\frac{\cosh x+\cos x}{6}\right\}=\frac{2}{6}=\frac{1}{3}$$

Atau…….

$$\lim_{x\to0}\left\{\frac{\sinh x-\sin x}{x^3}\right\}=\lim_{x\to0}\left\{x+\frac{\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)}{x^3}\right\}$$

$$=\lim_{x\to0}\left\{\frac{\frac{2x^3}{3!}+\frac{2x^7}{7!}}{x^3}\right\}$$

e. $\lim_{x\to1}\left\{\frac{x^3-2x^2+4x-3}{4x^2-5x+1}\right\}=\left\{\frac{1-2+4-3}{4-5+1}\right\}=\frac{0}{0}$

$$\lim_{x\to1}\left\{\frac{x^3-2x^2+4x-3}{4x^25x+1}\right\}=\lim_{x\to1}\left\{\frac{3x^2-4x+4}{8x-5}\right\}$$

$$\left\{\frac{3-4+4}{8-5}\right\}=\frac{3}{8}=1$$

f. $\lim_{x\to0}\left\{\frac{\tan x-x}{\sin x-x}\right\}=\lim_{x\to0}\left\{\frac{\sec^2 x-1}{\cos x-1}\right\}=\frac{1-1}{1-1}=0$

$$\lim_{x\to0}\left\{\frac{2\sec^2 x\,.\tan x}{-\sin x}\right\}=\frac{0}{0}$$

$$\lim_{x\to}\left\{\frac{2\sec^2 x\,.\sec^2+2\tan x\,.2\sec x\,.tgx}{-\cos x}\right\}=\frac{2}{-1}=-2$$

g. $\lim_{x\to0}\left\{\frac{x\cos x-\sin x}{x^3}\right\}$

DERET UKUR = $U_n = ar^{n-1}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Deret yang jumlah *n* sukunya (S_n) tidak mempunyai nilai limit, artinya deret divergen.

Contoh:

1. Tinjaulah deret ukur: $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$ konvergen atau divergen

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}, a = 1, r = \frac{1}{3}$$

$$S_n = \frac{1\left(1 - \left({}^{1}\!/_{3}\right)^n\right)}{1 - {}^{1}\!/_{3}} = \frac{1 - \frac{1}{3^n}}{\frac{2}{3}} = \frac{2}{3}\left(1 - \frac{1}{3^n}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{3}\left(1 - \frac{1}{3^n}\right) = \frac{2}{3} \qquad \text{(konvergen)}$$

2. Tinjaulah deret ukur: 1+3+9+27+81+... konvergen atau divergen

$$S_n = \frac{1(1 - 3^n)}{1 - 3} = \frac{1 - 3^n}{-2} = \frac{1 - 3^n}{-2} = \frac{3^n - 1}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3^n - 1}{2} = \infty \qquad \text{(divergen)}$$

DERET HITUNG = Un = a + (n-1) d

$$Sn = \frac{n}{2}(2a + (n-1)d)$$

DERET PANGKAT BILANGAN ASLI

$\sum_1^N r = \frac{n(n+1)}{2}$ $\quad \sum_1^n r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

$$\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

DERET TAK BERHINGGA : Sn = u1 + u2 + ... un

Jika $\underset{n \to \sim}{Lim} Sn$ memberikan harga tertentu $\rightarrow$ konvergen $S\sim = \frac{a}{1-r}$

$\underset{n \to \sim}{Lim} Sn$ tidak diberikan harga tertentu $\Rightarrow$ divergen.

1. $\underset{n \to \sim}{Lim} \frac{5n+3}{2n-7} = \underset{n \to \sim}{Lim} \frac{5+3/n}{2-7/n} = \frac{5}{2}$

2. $\underset{n \to \sim}{Lt} \frac{2n^2+4n-3}{5n-6n+1} = \underset{n \to \sim}{Lt} \frac{2+4/n-3/n^2}{5-6/n+1/n^2} = \frac{2}{5}$

3. $\underset{n \to \sim}{Lt} \frac{n^3-2}{2n^3+3/n^2-4/n^2} = \frac{1}{2}$

Cara 1

$\underset{n\to\sim}{Lt}\ S_n \to$ tertentu $\to$ konvergen

$\underset{n\to\sim}{Lt} = \frac{1(1-3)}{1-3} = \frac{1(1-\sim)}{-2} = -\frac{1}{2} - \sim = \sim divergen$

Cara 2

U_n $= c \to$ konvergen

$\underset{n\to\sim}{Lt}\ 1.3^{n-1}$ $= \sim \neq 0 \to$ divergen

Cara 3

$\underset{n\to\sim}{Lt}\ \frac{U_n+1}{U_n} = \underset{n\to\sim}{Lt}\ \frac{3^n}{3^{n-1}} = 3^{\frac{n}{n-1}} = 3^{n-n+1} = 3^1 = 3\ divergen$

U_n $= 1.3^{n-1}$

U_n+1 $= 3^{(n+1)-1} = 3^n$

DERET TAYLOR

Deret Taylor diperoleh saat f pada $x = a$

$$\boldsymbol{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \cdots + \frac{f^n(a)}{n!}.(x-a)^n + \cdots}$$

Contoh:

1. Tentukan Deret Taylor yang dihasilkan oleh fungsi $f(x) = \frac{1}{x}$ pada a=1

   Pertama, turunkan fungsi $f(x) = \frac{1}{x}$

   $f(x) = x^{-1} \to f(1) = 1$

   $f'(x) = -x^{-2} \to f'(1) = -1$

$f''(x) = 2x^{-3} \rightarrow f''(1) = 2 = 2!$

$f'''(x) = -6x^{-4} \rightarrow f'''(1) = -6 = -3!$

$f^{(4)}(x) = 24x^{-5} \rightarrow f^{(4)}(1) = 24 = 4!$

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \rightarrow f^{(n)}(1) = (-1)n!$

Langkah kedua, masukkan ke persamaan Deret Taylor

$f(x) = 1 - 1(x-1) + \frac{2!(x-1)^2}{2!} - \frac{3!(x-1)^3}{3!} + \frac{4!(x-1)^n}{3!} + (-1)^n \frac{n!}{x^{n+1}} - + \cdots$

2. Tentukan deret Taylor yang dihasilkan oleh *f(x)*=sin x pada x=(π/4).

   Penyelesaian:

   $f(x) = sin\ x \rightarrow f(\pi/4) = 1/2\sqrt{2}$

   $f'(x) = cos\ x \rightarrow f'(\pi/4) = 1/2\sqrt{2}$

   $f''(x) = -sin\ x \rightarrow f''(\pi/4) = -1/2\sqrt{2}$

   $f'''(x) = -cos\ x \rightarrow f''(\pi/4) = -1/2\sqrt{2}$

   $sin\ x\ = f\frac{\pi}{4} + \frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \cdots$

   $sin\ x\ = \frac{1}{2}\sqrt{2}\ + \frac{\frac{1}{2}\sqrt{2}}{1!}\left(x - \frac{\pi}{4}\right) + \frac{\frac{1}{2}\sqrt{2}}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{\frac{1}{2}\sqrt{2}}{3!}\left(x - \frac{\pi}{4}\right)^3 +$

   $\cdots\cdots \frac{\frac{1}{2}\sqrt{2}}{n!}\left(x - \frac{\pi}{4}\right)^n$

DERET MACLAURIN

Untuk a=0 Deret Taylor disebut Deret **Maclaurin.**

$$\boldsymbol{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^{2} + \frac{f'''(0)}{3!}x^{3} + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{(n-1)} + \cdots}$$

Contoh:

1. Carilah deret Maclaurin $f(x) = e^x$

   Penyelesaian:

   $f(x) = e^{x} \rightarrow f(0) = e^{0} = 1$

   $f'(x) = e^{x} \rightarrow f'(0) = e^{0} = 1$

   $f''(x) = e^{x} \rightarrow f''(0) = e^{0} = 1$

   $f'''(x) = e^{x} \rightarrow f'''(0) = e^{0} = 1$

   $f(x) = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots + \frac{x^{n}}{n!} + \cdots$

2. Tentukan deret Maclaurin dari *f(x) = sin x*

   Penyelesaian:

   $f(x) = sin\, x \rightarrow f(0) = 0$

   $f'(x) = cos\, x \rightarrow f'(0) = 1$

   $f''(x) = -sin\, x \rightarrow f''(0) = 0$

   $f'''(x) = -cos\, x \rightarrow f'''(0) = -1$

   $f^{(4)}(x) = sin\, x \rightarrow f^{(4)}(0) = 0$

   $sin\, x = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \frac{x^{7}}{7!} + \cdots$

KAIDAH UJI KEKONVERGENAN

1. Jika $\lim_{n\to\sim} Un = 0$ deret mungkin konvergen.

   Jika $\lim_{n\to\sim}$ Un $\neq 0$, deret pasti divergen.

   Uji perbandingan-deret pembanding yang penting

   $$\frac{1}{p^1}+\frac{1}{p^2}+\frac{1}{p^3}+\frac{1}{p^5}+\ldots+\frac{1}{p^n}$$

   Jika p>1 deret konvergen
   p<1 deret divergen.

   Contoh:

   a. Tentukan apakah deret $\sum_{n=1}^{\infty}\frac{n}{3^n}$ konvergen.

   $$\rho = \lim_{n\to\infty}\frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty}\frac{n+1}{3^{n+1}}\left(\frac{3^n}{n}\right) = \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{3^n}{3^{n+1}}\right) = \frac{1}{3}$$

   Karena nilai $\rho < 1$, maka deret konvergen.

   b. Untuk menguji kekonvergenan deret

   $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\frac{1}{6^6}+\ldots$ dibandingkan dengan deret

   $1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\ldots$ dibandingkan dengan suku seletak $\frac{1}{3^3}<\frac{1}{2^3}$; $\frac{1}{4^4}<\frac{1}{2^4}$; $\frac{1}{5^5}<\frac{1}{2^5}$ dan seterusnya.

   Terlihat setelah lewat dua suku pertama suku-suku deret pertama selalu lebih kecil daripada suku-suju seletak deret lain yang konvergen, maka deret tersebut adalah konvergen.

2. D Alembert untuk suku-suku positif

$\lim\limits_{n\to\sim} \frac{un+1}{Un} < 1 \Rightarrow$ konvergen

$\lim\limits_{n\to\sim} \frac{Un+1}{Un} > 1 \Rightarrow$ divergen

$\lim\limits_{n\to 2} \frac{Un+1}{Un} = 1 \Rightarrow$ tidak ada kesimpulan.

Contoh:

- $\frac{1}{1} + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^5} + \ldots.$

  $U_n \to$ atas DH $\to U_n = a + (n\text{-}1)d$

  $= 1 + (n\text{-}1)2$

  $= 1 + (2n\text{-}2)$

  $U_n = 2n\text{-}1$

  $U_n \to$ bawah DU $\to U_n = ar^{n-1}$

  $a = 1 = 1.2^{n-1}$

  $r = 2 = 2^{\ n-1}$

  $U_n \frac{2n-1}{2^{n-1}}, U_n + 1 = \frac{2(n+1)}{2^{(n+1)-1}} = \frac{2n+1}{2^n}$

  Cara D'Albert

$$\lim\limits_{n\to\sim} \frac{U_n+1}{U_n} = \frac{2n+1/n^n}{2n-1/2^{n-1}} = \lim\limits_{n\to\sim} \frac{2n+1}{2^n} = \frac{2^{n-1}}{2n-1}$$

$$= \lim\limits_{n\to\sim} \frac{2n+1}{2n-1} . \frac{2^{n-1}}{2n-1}$$

$$= \lim\limits_{n\to\sim} \frac{2n+1}{2n-1} . 2^{n-n-1}$$

$$= \lim\limits_{n\to\sim} \frac{2n+1}{2n-1} . \frac{1}{2}$$

$$= \underset{n \to \sim}{Lt} \frac{2 + {}^{1}/_{2}}{2 - {}^{1}/_{n}} \cdot \frac{1}{2}$$

= ½ konvergen

- $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \ldots..$

  U_n atas = 1 + (n-1) 1 = 1 + n – 1 = n

  U_n bawah = 2 + (n-1) = 2 +n-1 = n + 1

  $U_n = \frac{n}{n+1}, U_n + 1 = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$

  a) D'Alembert

$$\underset{n \to \sim}{Lt} \frac{U_n + 1}{U_n} = \frac{n + 1/n + 2}{n/n + 1}$$

$$\underset{n \to \sim}{Lt} \frac{n + 1}{n + 2} \cdot \frac{n + 1}{n}$$

$$\underset{n \to \sim}{Lt} \frac{n^2 + 2n + 1}{n^2 + 2n}$$

$$\underset{n \to \sim}{Lt} \frac{1 + {}^{2}/_{n} + {}^{2}/_{n^2}}{n^2 + 2n}$$

$$\underset{n \to \sim}{Lt} \frac{1 + {}^{2}/_{n} + {}^{2}/_{n^2}}{1 + {}^{2}/_{n}}$$

= tidak ada kesimpulan

b) $\underset{n\to\sim}{Lt}\ U_n = \underset{n\to\sim}{Lt}\ \frac{n}{n+1} = \frac{1}{1+{}^{1}/_{n}}$=1 divergen

- $\frac{1}{5}+\frac{2}{6}+\frac{2^2}{7}+\frac{2^3}{8}+\ldots.$

  atas U_n $= ar^{n-1} = 1.2^{n-1} = 2^{n-1}$

  bawah U_n $= 5 + (n\text{-}1)\ 1 = 5 + n - 1 = 4 + n$

  $U_n = \frac{2^{n-1}}{4+n}, U_n + 1 = \frac{2^{n+1-n}}{4+n+1} = \frac{2}{5+n}$

  D'Alembert

  $$\underset{n\to\sim}{Lt} \frac{2^n}{5+n} \Big/ \frac{2^{n-1}}{4+n}$$

  $$\underset{n\to\sim}{Lt} \frac{2^2}{5+n} . \frac{4+n}{2^{n-1}}$$

  $\underset{n\to\sim}{Lt}\ 2^{n-n+1} . \frac{4+n}{5+n}$

  $\underset{n\to\sim}{Lt}\ 2^1 . \frac{(4+n)}{5+n}$

  $= \frac{8/n+2}{5/n+1}$

  = 2 divergen

3. Konvergen Mutlak (*Absolutely Convergent*)
   Suatu deret yang suku-sukunya positif dan negative (*Alternating Series*).

   Deret : $\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +}$ (konvergen)

   Deret : $\mathbf{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +}$ (divergen)

Jika $\sum|Un|$ konvergen, $\sum un$, konvergen mutlak.
Jika $\sum|Un|$ divergen, tetapi $\sum Un$, konvergen maka $\sum Un$ konvergen bersyarat.

Jadi jika $\sum \mathbf{U_n} = \mathbf{1 - 3 + 5 - 7 + 9 + \ldots}$
maka $\sum |\mathbf{U_n}| = \mathbf{1 + 3 + 5 + 7 + 9 + \ldots}$
Jika $\sum |\mathbf{U_n}|$ konvergen maka deret $\sum \mathbf{U_n}$ dikatakan konvergen mutlak
Jika $\sum |\mathbf{U_n}|$ divergen tetapi $\sum \mathbf{U_n}$ konvergen, maka $\sum \mathbf{U_n}$ dikatakan sebagai konvergen bersyarat.

Contoh: Tentukanlah daerah harga x dimana deret berikut konvergen mutlak.

$$\frac{x}{2{,}5} - \frac{x^2}{3{,}5^2} + \frac{x^3}{3{,}5^3} - \frac{x^4}{3{,}5^4} + \frac{x^5}{3{,}5^5} - \ldots$$

Penyelesaian:

$$|u_n| = \frac{x^n}{(n+1)5^n};\ |u_n + 1| = \frac{x^{n+1}}{(n+1)5^{n+1}}$$

$$\frac{|u_n + 1|}{|u_n|} = \frac{x^{n+1}}{(n+1)5^{n+1}} \frac{(n+1)5^n}{x^n} = \frac{x(n+1)}{5(n+2)} = \frac{x(1+1/n)}{5(n+2/n)}$$

$\lim\limits_{n\to\infty} \frac{|u_n+1|}{|u_n|} = \frac{x}{5}$, supaya konvergen mutlak maka $\lim\limits_{n\to\infty} \frac{|u_n+1|}{|u_n|} < 1$

Deret tersebut konvergen mutlak jika maka $\left|\frac{x}{5}\right| < 1$, yaitu $|x| < 5$.

CONTOH DERET PANGKAT

1. $\text{Sin x} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{5!} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots -\infty < x < \infty$

2. $\text{Cos x} = 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots -\infty < x < \infty$

3. $\text{Tan x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots.$

4. $(1+x)^n = 1 + xn + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{n(n-1)...(n-x+1)x^n}{n!} + \cdots$

5. $(1+n)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n+1)x^3}{3!} + \ldots.$

6. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots -1 \leq x \leq 1$

7. $\text{Sin}^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$

8. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{5!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \quad -\infty < x < \infty$

9. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$

10. $\text{Sinh x} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$

11. $\text{Cosh x} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots.$

12. $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots (-1)^{n-1}\frac{x^n}{n} + \cdots \quad -1 < x \leq 1$

13. $\frac{1}{2} ln\left|\frac{1+x}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \quad -1 < x \leq 1$

14. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad -1 < x < 1$

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PROFIL PENULIS

Dr. Ir. Siti Jamilatun, M.T. adalah dosen tetap Fakultas Teknologi Industri (FTI) Universitas Ahmad Dahlan sejak tahun 1996 sampai sekarang. Penulis mengajar baik di Program S-1 Teknik Kimia maupun Program Magister Teknik Kimia di lingkungan FTI UAD. Penulis mendapatkan gelar Insinyur (Ir) tahun 1991, Magister Teknik (M.T.) tahun 2002 dan Doktor tahun 2019 dari Fakultas Teknik (FT) Universitas Gadjah Mada (UGM) jurusan Teknik Kimia, serta gelar Insinyur dari Program Profesi Insinyur Universitas Muslim Indonesia tahun 2022.

Selain mengajar, penulis juga aktif melakukan penelitian dan Pengabdian. Penulis adalah peneliti senior di Universitas Ahmad Dahlan dengan H index scopus 6, H index Sinta 12. Jabatan yang pernah dijabat oleh penulis adalah sebagai Ketua Program Studi Teknik Kimia (2012-2014), dan pada saat edisi ini terbit penulis menjabat sebagai Ketua Program Studi Magister Teknik Kimia UAD.

Bidang keilmuan yang menjadi minat utamanya adalah Pengolahan Biomassa, Biorefinary dan Renewable Energy. Saat ini topik tentang pengolahan biomassa dengan pirolisis telah menghasilkan paper di jurnal internasional terindex scopus dan terindex Sinta.

DIKTAT MATA KULIAH

KALKULUS

Diktat Kalkulus ini bertujuan untuk meningkatkan kemampuan mahasiswa dalam mengembangkan konsep/ teori dalam bidang keteknikkimiaan. Kajian utama meliputi: Sistem bilangan, Fungsi , Limit dan kontinuitas, Turunan (differential), Deret sederhana dan deret pangkat, Integral, Fungsi Gamma dan Beta. Perkuliahan dilaksanakan dengan pendekatan student center learning. Penilaian berbasis kompetensi mencakup: partisipasi aktif, portofolio tugas-tugas, dan ujian kompetensi.

Diktat ini membahas teori, soal dan penyelesaian tentang

1. Sistem bilangan
2. Fungsi
3. Limit dan kontinuitas
4. Deret sederhana dan pangkat
5. Fungsi gamma dan fungsi beta
6. Turunan (diferensial)
7. Integral

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