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Article List [more]

1

Title: *Some Results on the Higher Multiplier of a Pair of Groups*
Vol.45(4) (2021) page: 429-435
Author(s): H. Arabyani
Abstract:

In this paper, we investigate the notion of the c-nilpotent multiplier of a pair of groups and present some exact sequences and isomorphisms for the c-nilpotent multiplier of a pair of groups. Also, we provide a sufficient condition for the c-nilpotent multiplier of a pair of groups to be finite. Moreover, we give some conditions under which the c-nilpotent multiplier of a pair of groups is not trivial.

Keywords: Pair of groups; Nilpotent group; Schur multiplier.

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Title: *Initial Coefficients of Certain Sub-classes of Bi-univalent Analytic Function on the Exterior of the Unit Disc*
Vol.45(4) (2021) page: 437-443
Author(s): S. Barik and A. Kumar Mishra
Abstract:

We develop a new method to find improved bounds on the modulii of the $zero^{th}$, first and second coefficients for the functions in certain sub-classes of bi-univalent functions in the exterior of the unit disc of the complex plane. Our bounds are obtained by refining well known estimates for the initial coefficients of the Carthéodory functions.

Keywords: Analytic function; Analytic continuation; Univalent functions; Bi-univalent functions; Coefficient bounds.

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Title: *Path Integrals for Scalar Fields: The Henstock Approach*
Vol.45(4) (2021) page: 445-459
Author(s): V. Boonpogkrong
Abstract:

In the classical physics, the quantum mechanics can be formulated in terms of path integrals. A path integral is an operator calculus, as pointed out by Feynman. The gauge approach to path integrals was first presented by Henstock in 1970's, see [5]. This approach was elaborated further in [8]. In this paper, we present a path integral for scalar fields $\phi(x,t)$, where $x, t \in [0,\infty)$, using the gauge approach called Henstock approach in this paper. The equivalent theorem between the integral defined by using Henstock approach and the classical definition is proved. The sequentially equivalent definitions are discussed.

Keywords: Henstock integral; Path integral; Feynman integral; Operator calculus; Field theory; Wiener measure.

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Title: *Adaptive Sampling Recovery and Nonlinear Approximations of Multivariate Functions in Besov-type Spaces*
Vol.45(4) (2021) page: 461-482
Author(s): N.M. Cuong
Abstract:

We investigate nonlinear approximations by sets of finite cardinality or of finite pseudo-dimesion and the optimality in terms of entropy numbers and other characterizations for nonlinear approximations. Functions to be approximated are in Besov type spaces of functions having a certain mixed smoothness. We prove the asymptotic order of these quantities and explicitly construct asymptotically optimal methods of nonlinear approximation based on a trigonometric sampling representation in Besov type spaces.

Keywords: Besov-type spaces; Sampling trigonometric representation; Nonlinear approximation; Adaptive sampling recovery; Entropy numbers; pseudo-dimension.

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Title: *A Polynomial Shared by an Entire Function and its Linear Differential Polynomials*
Vol.45(4) (2021) page: 483-495
Author(s): G.K. Ghosh
Abstract:

The uniqueness problems on entire functions sharing at least two values with their derivatives have been studied and many results on this topic have been obtained. In this paper, we study an entire function $f(z)$ that shares a polynomial $a(z)$ with $f^{(1)}(z)$, together with higher order derivatives of linear differential polynomials generated by them.

Keywords: Entire function; Polynomial; Linear differential polynomials.

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Title: *Inequalities for the Derivative of a Polynomial*
Vol.45(4) (2021) page: 497-508
Author(s): M.H. Gulzar, B.A. Zargar and R. Akhter
Abstract:

In this paper, we find a lower bound for the maximum moduli of the derivative and polar derivative of a polynomial in terms of the moduli of the coefficients and the maximum modulus and the minimum modulus of the polynomial. Our paper generalises and also refines many known results in the field.

Keywords: Polynomial; Polar derivative; Inequalities.

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Title: *On Golden Semi-Symmetric Non-Metric F-Connections*
Vol.45(4) (2021) page: 509-520
Author(s): C. Karaman and A. Gezer
Abstract:

The main purpose of the present paper is to construct golden semi-symmetric non-metric F-connections on a locally decomposable golden Riemannian manifold and investigate some properties of torsion and curvature tensors of these connections.

Keywords: Golden Riemannian structure; Dual connection; Semi-symmetric non-metric F-connection; Tachibana operator.

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Title: *On Cyclic Codes with Minimal Generating Sets over the Ring* $\mathbb{Z}_{p^k}$
Vol.45(4) (2021) page: 521-532
Author(s): O.M. Prakash, H. Islam and S. Das
Abstract:

For a prime p and an integer $k > 1$, $\mathbb{Z}_{p^k}$ denotes the ring of residue classes of integers modulo p^k. This article completely determines the structure of cyclic codes over $\mathbb{Z}_{p^k}$. Also, minimal generating sets for those codes are obtained. Finally, some computational examples are given in support of our results.

Keywords: Cyclic code; Hamming distance; Generator polynomial.

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Title: *Existence and Concentration of Ground State Solutions for the Coupled Nonlinear Schr*

Vol.45(4) (2021) page: 533-560
Author(s): W.B. Wang, Q.Q. Li, J.W. Zhou and Y.K. Li
Abstract:

This article concerns the coupled Schr\"{o}dinger system in whole space $\mathbb{R}^N$

$$

-\varepsilon^{2}\Delta u+V(x)u=P(x)|u|^{p-2}u+Q(x)|u|^{\frac{p}{2}-2}u|v|^{\frac{p}{2}},

$$

$$

-\varepsilon^{2}\Delta v+V(x)v=Q(x)|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}v+P(x)|v|^{p-2}v,\\

$$

where $\varepsilon > 0$ is a small parameter. Under some suitable conditions, we obtain the existence of ground state for $\varepsilon > 0$ via the Nehari manifold and concentration-compactness principles. Furthermore, we prove these ground state solutions concentrate at some set related to V, P and Q.

Keywords: Coupled Schr\

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Title: *On Jointly Second (R,S)-Submodules*
Vol.45(4) (2021) page: 561-570
Author(s): D.A. Yuwaningsih and I.E. Wijayanti
Abstract:

Let R and S be commutative rings and M be an (R, S)-module. In this paper, we present the dual notion of jointly prime (R, S)-submodules, that is called jointly second (R, S)-submodules, and we investigate some properties of them. We give a necessary and sufficient condition for an (R, S)-submodule being jointly second (R, S)-submodules. Moreover, we present the definition of jointly second (R, S)-modules and present a condition for jointly prime (R, S)-modules being jointly second (R, S)-modules and vice versa.

Keywords: Second submodules; Coprime submodules; Jointly prime; Second modules.

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Title: *Call for Papers*
Vol.45(4) (2021) page:
Author(s):
Abstract:
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Southeast Asian Bulletin of Mathematics (2021) 45: 561–570

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On Jointly Second (R, S)-Submodules

Dian Ariesta Yuwaningsih
Department of Mathematics Education, Universitas Ahmad Dahlan, Indonesia
Email: dian.ariesta@pmat.uad.ac.id

Indah Emilia Wijayanti
Department of Mathematics, Universitas Gadjah Mada, Indonesia
Email: ind_wijayanti@ugm.ac.id

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Abstract. Let R and S be commutative rings and M be an (R, S)-module. In this paper, we present the dual notion of jointly prime (R, S)-submodules, that is called jointly second (R, S)-submodules, and we investigate some properties of them. We give a necessary and sufficient condition for an (R, S)-submodule being jointly second (R, S)-submodules. Moreover, we present the definition of jointly second (R, S)-modules and present a condition for jointly prime (R, S)-modules being jointly second (R, S)-modules and vice versa.

Keywords: Second submodules; Coprime submodules; Jointly prime; Second modules.

1. Preliminaries

Throughout this article, R and S will denote commutative rings and M be an additive Abelian group. Furthermore, $\mathbb{Z}$ will denote the ring of integers.

Let M be a left R-module. Based on [5], a proper submodule N of M is called prime if for each $r \in R$ and $m \in M$ such that $rm \in N$ implies either $m \in M$ or $rM \subseteq N$. This definition has been generalized by [9]. Based on [9], a proper submodule N of M is called prime if for each $r \in R$, the homomorphism $g_r : M/N \to M/N$ is either injective or zero. M is said to be second modules if the zero submodule of M is prime.

Moreover, [9] also introduced the dual notion of prime submodules, that is called second submodules. A non-zero submodule N of M is said to be second if for each $r \in R$, the homomorphism $f_r : N \to N$ is either surjective or zero. Further, [9] also provided some properties of second submodules. Several researchers have studied this second submodule, among of them are [1], [2], [3], [4], [8], and [6].

On the other hand, [7] defined the structure (R, S)-modules as a generalization of (R, S)-bimodules. Further, [7] also introduced the definition of jointly prime (R, S)-submodule, when R and S are arbitrary rings (not necessary commutative). A proper (R, S)-submodule P of M is called jointly prime if for any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. When R and S are commutative rings, we have a proper (R, S)-submodule P of M is called jointly prime if for any ideal I of R, ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. Moreover, the research about (R, S)-modules have been developed in [10] and [11].

The main purpose is to provide some information concerning the dual notion of jointly prime (R, S)-submodules. We will call this dual notion as jointly second (R, S)-submodules.

In Section 2, we give the definition of jointly second (R, S)-submodules and some examples of them. Moreover, we also provide some properties of jointly second (R, S)-submodules. Among of them are the necessary and sufficient condition for an (R, S)-submodule being jointly second; every simple (R, S)-submodule is jointly second; the annihilator of jointly second (R, S)-submodules is prime; and every jointly second (R, S)-submodule contained in maximal jointly second (R, S)-submodule.

In [9], a left R-module M is called second if M is second submodules for itself. In Section 3, we give the definition of jointly second (R, S)-modules. Moreover, we provide some properties of them. At the end, we present a condition for jointly prime (R, S)-modules being jointly second (R, S)-modules and vice versa.

2. Some Properties of Jointly Second (R, S)-Submodules

In this section we present the definition of jointly second (R, S)-submodule and some properties of them. However, it should be noted earlier the definition of jointly prime (R, S)-submodules, which was introduced by [7] as follow.

Definition 2.1. *Let R and S be arbitrary rings. A proper (R, S)-submodule P of M is called jointly prime if for any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$.*

When R and S are commutative rings, we have the definition of jointly prime (R, S)-submodule as follows.

Definition 2.2. *A proper (R, S)-submodule P of M is called jointly prime (R, S)-submodule if for any ideal I of R, ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$.*

The definition of jointly second (R, S)-submodule is given as follows.

Definition 2.3. *An (R, S)-submodule N of M is called jointly second (R, S)-submodule if $N \neq 0$ and for each $r \in R$, the homomorphism (R, S)-module $f_r : N \longrightarrow N$ with definition $f_r(n) = rnS$ for each $n \in N$, is an epimorphism or zero homomorphism.*

Note that if f_r is an epimorphism, we have $Im(f_r) = N$. However, if f_r is a zero homomorphism then for each $n \in N$ satisfies $f_r(n) = 0$, so $rnS = 0$. Because it applies to every $n \in N$, then we have $rNS = 0$.

Next, we give an example of jointly second (R, S)-submodule.

Example 2.4. Consider the $(\mathbb{Z}, \mathbb{Z})$-module $\mathbb{Z}_{12}$.

(1) The $(\mathbb{Z}, \mathbb{Z})$-submodule $N = \{\bar{0}, \bar{6}\}$ of $\mathbb{Z}_{12}$ is a jointly second $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Z}_{12}$. For $m \in \mathbb{Z}$, we construct an $(\mathbb{Z}, \mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : N &\longrightarrow N \\ \bar{n} &\longmapsto f_m(\bar{n}) = \overline{mn\mathbb{Z}} \quad , \forall \bar{n} \in N. \end{aligned}$$

If $m \in 2\mathbb{Z}$, then we obtain $f_m(N) = \{\bar{0}\}$.
If $m \notin 2\mathbb{Z}$, then we have $f_m(N) = N$.
Thus, f_m is a zero homomorphism or an epimorphism. Hence, it is proved that N is a jointly second $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Z}_{12}$.

(2) The $(\mathbb{Z}, \mathbb{Z})$-submodule $K = \{\bar{0}, \bar{4}, \bar{8}\}$ of $\mathbb{Z}_{12}$ is a jointly second $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Z}_{12}$. For $m \in \mathbb{Z}_{12}$, we construct an $(\mathbb{Z}, \mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : K &\longrightarrow K \\ \bar{k} &\longmapsto f_m(\bar{k}) = \overline{mk\mathbb{Z}} \quad , \forall \bar{k} \in N. \end{aligned}$$

If $m \in 3\mathbb{Z}$, then we have $f_m(K) = \{\bar{0}\}$.
If $m \notin 3\mathbb{Z}$, then we get $f_m(K) = K$.
Thus, f_m is a zero homomorphism or an epimorphism. Hence, it is proved that K is a jointly second $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Z}_{12}$.

Now, we give an example of (R, S)-submodule which is not jointly second.

Example 2.5. Let $2\mathbb{Z}$ be an $(\mathbb{Z},\mathbb{Z})$-module. An $(\mathbb{Z},\mathbb{Z})$-submodule $4\mathbb{Z}$ of $2\mathbb{Z}$ is not a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $2\mathbb{Z}$. For any element $m \in \mathbb{Z}$, we construct an $(\mathbb{Z},\mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : 4\mathbb{Z} &\longrightarrow 4\mathbb{Z} \\ a &\longmapsto f_m(a) = ma\mathbb{Z} \quad ,\forall a \in 4\mathbb{Z}. \end{aligned}$$

If $m = 0$, then we have $f_m(4\mathbb{Z}) = \{0\}$.

But, if $m \neq 0$ then not necessary $f_m(4\mathbb{Z}) = 4\mathbb{Z}$.

For the example, let any element $m = 2 \in \mathbb{Z}$. Then we have:

$$f_m(4\mathbb{Z}) = 2(4\mathbb{Z})\mathbb{Z} \subseteq 8\mathbb{Z} \subseteq 4\mathbb{Z}$$

but $f_m(4\mathbb{Z}) \neq 4\mathbb{Z}$. So, we get $4\mathbb{Z}$ is not jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $2\mathbb{Z}$.

According to [7], for each (R,S)-submodule N of M, let the set

$$(K :_R M) = \{r \in R \mid rMS \subseteq K\}.$$

In general $(K :_R M)$ is only an additive subgroup of R. But if we have the condition $S^2 = S$, clearly that $(K :_R M)$ is an ideal of R. We may also say that $(K :_R M)$ is the annihilator of quotient (R,S)-module M/K over the ring R.

Now, we present some properties of jointly second (R,S)-submodules.

Proposition 2.6. *Let M be an (R,S)-module with $S^2 = S$. An (R,S)-submodule N of M is a jointly second (R,S)-submodule if and only if $(0 :_R N) = (K :_R N)$ for each proper (R,S)-submodule K of N.*

Proof. $(\Rightarrow)$. Let N be jointly second (R,S)-submodules of M. Then, $N \neq 0$ and for each $r \in R$, the (R,S)-module homomorphisms

$$\begin{aligned} f_r : N &\longrightarrow N \\ n &\longmapsto f_r(n) = rnS, \quad \forall n \in N \end{aligned}$$

is an epimorphism or zero homomorphism. Let any $x \in (K :_R N)$. Then $xNS \subseteq K \subset N$. If the homomorphism f_x is an epimorphism then $f_X(N) = N$, so we get $xNS = N \subseteq K$. A contradiction with $K \subset N$. So, f_x is a zero homomorphism. Thus, we obtain $f_x(N) = 0$ or $xNS = 0$, so $x \in (0 :_R N)$. Thus, we obtain $(K :_R N) \subseteq (0 :_R N)$. Moreover, let any $y \in (0 :_R N)$. Then $yNS = 0$. Since K is an (R,S)-submodule of M, $yNS = 0 \subseteq K$. Thus, $y \in (K :_R N)$. Hence, we obtain $(0 :_R N) \subseteq (K :_R N)$. Thus, it is proved that $(0 :_R N) = (K :_R N)$.

$(\Leftarrow)$. Let any proper (R,S)-submodule K of N satisfy $(0 :_R N) = (K :_R N)$. For any $r \in R$, we construct an (R,S)-module homomorphism

$$\begin{aligned} f_r : N &\longrightarrow N \\ n &\longmapsto f_r(n) = rnS, \quad \forall n \in N. \end{aligned}$$

Assume that f_r is not an epimorphism. We will show that f_r is a zero homomorphism. Since f_r not an epimorphism, $Im(f_r) \neq N$, so that $rNS \neq N$. Suppose $Im(f_r) = K$. Then $rNS = K$, so that $r \in (K :_R N)$. Since $(0 :_R N) = (K :_R N)$, $r \in (0 :_R N)$, so that $rNS = 0$. Consequently, we get $f_r(N) = 0$. Thus, it is proved that f_r is a zero homomorphisms. Hence, N is a jointly second (R,S)-submodule of M. ■

An (R,S)-module M is said to be simple if the (R,S)-submodule of M is only zero submodule and M itself.

Proposition 2.7. *Let M be an (R,S)-module with $S^2 = S$ and N a simple (R,S)-submodule of M. Then, N is a jointly second (R,S)-submodule of M.*

Proof. It is known that N is a simple (R,S)-submodule of M, so N only contains (R,S)-submodule $\{0\}$ and N itself. Since $\{0\}$ is the only proper (R,S)-submodule of N, we can form (R,S)-module factor $N/\{0\} = N$. Based on Proposition 2.6, we have $(0 :_R N) = (\{0\} :_R N)$. Thus, N is a jointly second (R,S)-submodule of M. ■

Proposition 2.8. *Let M be an (R,S)-module with $S^2 = S$ and (R,S)-submodule N of M with $N \neq 0$. Then the following statements are equivalent:*

(1) *N is a jointly second (R,S)-submodule of M.*
(2) *For any ideal A of R, $ANS = 0$ or $ANS = N$.*
(3) *$ANS = N$ for any ideal A of R not contained in $(0 :_R N)$.*
(4) *$ANS = N$ for any ideal A of R properly containing $(0 :_R N)$.*

Proof. (1) $\Rightarrow$ (2). It is known that N is a jointly second (R,S)-submodule, meaning it fulfills $(0 :_R N) = (K :_R N)$ for each proper (R,S)-submodule K of N. We will show that $ANS = 0$ or $ANS = N$. Let any ideal A of R and assume that $ANS \neq N$. We will show that $ANS = 0$. Since $ANS \neq N$, ANS is a proper (R,S)-submodule of N, so that we can form (R,S)-module factor N/ANS. Suppose that $B = (0 :_R N/ANS)$. Then, we have

$$BNS = \Big\{ \sum_{i=1}^{k} b_i n_i S \mid b_i \in B,\ n_i \in N \Big\}.$$

Since $B = (0 :_R N/ANS)$, $BNS \subseteq ANS$, so that for each $s \in S$ satisfy $bns = a'n's'$ where $a' \in A$, $b \in B$, $n, n' \in N$, and $s' \in S$. As a result, we obtain

$$BNS = \Big\{ \sum_{j=1}^{l} a_j n_j S \mid a_j \in A,\ n_j \in N \Big\} = ANS.$$

Since N is a jointly second, we obtain $B = (0 :_R N/ANS) = (0 :_R N)$, so that $BNS = 0$. Thus, it is proved that $ANS = 0$.

(2) $\Rightarrow$ (3). Let any ideal A of R with $A \nsubseteq (0 :_R N)$. Then, $ANS \neq 0$. Based on the hypothesis we get $ANS = N$.

(3) $\Rightarrow$ (4). Let any ideal A of R with $(0 :_R N) \subset A$. That means $A \nsubseteq (0 :_R N)$. Based on the hypothesis we have $ANS = N$.

(4) $\Rightarrow$ (1). Let any proper (R, S)-submodule K of N. Suppose that $X = (K :_R N)$. We Have $(0 :_R N) \subseteq X$ and $XNS \subseteq K \neq N$ (since $(0 :_R N)$ is not proper subset of X). Because the only one that satisfied $N \neq XNS$ and $XNS \subseteq K$ is $XNS = 0$, we have $X = (0 :_R N)$. So, $(0 :_R N) = (K :_R N)$ for each proper (R, S)-submodule K of N. Hence, it is proved that K is a jointly second (R, S)-submodule of M. ■

Proposition 2.9. *Let M be an (R, S)-module with $S^2 = S$ and N be jointly second (R, S)-submodule of M. Then, $(0 :_R N)$ is a prime ideal of R.*

Proof. Let any ideal I and J of R such that $IJ \subseteq (0 :_R N)$. We will show that either $I \subseteq (0 :_R N)$ or $J \subseteq (0 :_R N)$. Since $IJ \subseteq (0 :_R N)$, we have $IJNS = 0$. Since N is a jointly second (R, S)-submodule, we have either $JNS = 0$ or $JNS = N$. If $JNS = N$, then $INS = IJNSS \subseteq IJNS = 0$, so that $INS = 0$. From here, we get $I \subseteq (0 :_R N)$. Moreover, if $JNS \neq N$ then $JNS = 0$, so that $J \subseteq (0 :_R N)$. Thus, we have either $I \subseteq (0 :_R N)$ or $J \subseteq (0 :_R N)$. Hence, $(0 :_R N)$ is a prime ideal of R. ■

Proposition 2.10. *Let M be an (R, S)-module with $S^2 = S$ and $(N_i)_{i \in I}$ be a chain of jointly second (R, S)-submodule of M. Then $N = \bigcup_{i \in I} N_i$ is jointly second (R, S)-submodule of M.*

Proof. Since $(N_i)_{i \in I}$ is a chain of jointly second (R, S)-submodule of M, $N = \bigcup_{i \in I} N_i$ is a non-zero (R, S)-submodule of M. Suppose formed $P_i = (0 :_R N_i)$ for each $i \in I$. Let any $i, j \in I$. Then $N_i \subseteq N_j$ or $N_j \subseteq N_i$, so we get either $P_j \subseteq P_i$ or $P_i \subseteq P_j$. Moreover, let any ideal A of R with $ANS \neq 0$. We will show that $ANS = N$. Since $ANS \neq 0$, there exist $n \in N_k$, $a \in A$, and $k \in I$ such that $anS \neq 0$. Consequently, we have $AN_kS \neq 0$. Since N_k is a jointly second (R, S)-submodule, $AN_kS = N_k \subseteq ANS$. If $P_i \subseteq P_k$ and $AN_kS \neq 0$ then $A \nsubseteq P_k$. As a result, $A \nsubseteq P_i$ so we have $N_i = AN_iS \subseteq ANS$. If $P_k \subseteq P_i$, that means $N_i \subseteq N_k$. Since $N_k = AN_kS \subseteq ANS$, we have $N_i \subseteq ANS$. Thus, we have $N_i \subseteq ANS$ for each $i \in I$. Clearly that $ANS \subseteq N$, so $ANS = N$. Hence, $N = \bigcup_{i \in I} N_i$ is jointly second (R, S)-submodule of M. ■

Proposition 2.11. *Let M be a non-zero (R, S)-module with $S^2 = S$. Then every jointly second (R, S)-submodule of M contained in maximal jointly second (R, S)-submodule of M.*

Proof. Let N be jointly second (R, S)-submodule of M. We construct the set $\mathfrak{J} = \{P \mid P$ jointly second (R, S)-submodule of M with $N \subseteq P\}$. It is obviously that $\mathfrak{J} \neq \emptyset$ since $N \in \mathfrak{J}$. By using Zorn's Lemma, we will show that $\mathfrak{J}$ has a maximal element. Equivalently showing that every non-empty chain $\mathfrak{C}$ of $\mathfrak{J}$ has an upper bound in $\mathfrak{J}$. Let any non-empty chain $\mathfrak{C} \in \mathfrak{J}$ and form the set $Q = \bigcup_{K\in\mathfrak{C}} K$. Based on Proposition 2.10, Q is also jointly second (R, S)-submodule of M. Since $N \subseteq Q$, $Q \in \mathfrak{J}$ and Q is an upper bound of $\mathfrak{C}$. Thus, it is proved that every non-empty chain $\mathfrak{J}$ has an upper bound in $\mathfrak{J}$. Therefore, based on Zorn's Lemma there exist a jointly second (R, S)-submodule $N^* \in \mathfrak{J}$ that maximal between all jointly second (R, S)-submodules of $\mathfrak{J}$. Thus, it is proved that every jointly second (R, S)-submodule N contained in maximal jointly second (R, S)-submodule N^* of M. ■

3. Jointly Second (R, S)-Modules

In this section, we present the definition of jointly second (R, S)-module and their properties. The definition of jointly second (R, S)-modules is given below.

Definition 3.1. *An (R, S)-module M is called a jointly second (R, S)-module if M is a jointly second (R, S)-submodule for itself.*

Now, we give some properties of jointly second (R, S)-modules. These properties based on the properties of jointly second (R, S)-submodule which was presented in the previous section.

Proposition 3.2. *Let M be a non-zero (R, S)-module with $S^2 = S$. Then, M is a jointly second (R, S)-module if $(0 :_R M) = (N :_R M)$ for every proper (R, S)-submodule N of M.*

Proof. Obviously from Proposition 2.6. ■

Proposition 3.3. *Let M be a non-zero (R, S)-module with $S^2 = S$. The following statements are equivalent:*

(1) *M is a jointly second (R, S)-module.*
(2) *For any ideal A of R, $AMS = 0$ or $AMS = M$.*
(3) *$AMS = M$ for any ideal A of R not contained in $(0 :_R M)$.*
(4) *$AMS = M$ for any ideal A of R properly containing $(0 :_R M)$.*

Proof. Clearly from Proposition 2.8. ■

Proposition 3.4. *Let M be an (R, S)-module with $S^2 = S$ and N be jointly second (R, S)-submodule of M. Then, $(0 :_R M)$ is a prime ideal of R.*

Proof. Evidently from Proposition 2.9. ■

Proposition 3.5. *Let M be a simple (R, S)-module with $S^2 = S$. Then, M is a jointly second (R, S)-module.*

Proof. Obviously from Proposition 2.7. ∎

Before proceeding to the next properties of jointly second (R,S)-modules, the following is given one of the properties of (R,S)-modules. This properties will be used in proving the necessary and sufficient conditions of jointly second (R,S)-module.

Let M be an (R,S)-module M and I an ideal of R that satisfy $I \subseteq Ann_R(M)$. We defined the scalar multiplication operation:

$$\begin{aligned} \_ \cdot \_ * \_ : R/I \times M \times S &\longrightarrow M \\ (\bar{a}, m, s) &\longrightarrow \bar{a} \cdot m * s := ams \end{aligned}$$

for each $\bar{a} \in R/I$, $m \in M$, and $s \in S$. Clearly that this scalar multiplication operation is closed. Moreover, we can show that this scalar multiplication operation is well-defined. Let any $\bar{a}, \bar{a'} \in R/I$, $m, m' \in M$, and $s, s' \in S$ with $(\bar{a}, m, s) = (\bar{a'}, m', s')$. This means that $\bar{a} = \bar{a'}$, $m = m'$, and $s = s'$. Since $\bar{a} = \bar{a'}$, we have $a - a' \in I$. Since $I \subseteq Ann_R(M)$, $(a-a')ms = 0$ so $ams = a'ms = a'm's'$. Thus, we have $\bar{a} \cdot m * s = \bar{a'} \cdot m' * s'$. Hence, it is proved that this scalar multiplication operation is well-defined.

Furthermore, we will show that an (R,S)-module M is an $(R/I,S)$-module over the scalar multiplication operation which is defined above. Let any $\bar{a}, \bar{a'} \in R/I$, $m, n \in M$, and $s, s' \in S$. Then we have:

(1) $\bar{a} \cdot (m+n) * s = a(m+n)s = ams + ans = \bar{a} \cdot m * s + \bar{a} \cdot n * s$.
(2) $(\bar{a} + \bar{a'}) \cdot m * s = (\overline{a + a'}) \cdot m * s = (a + a')ms = ams + a'ms = \bar{a} \cdot m * s + \bar{a'} \cdot m * s$.
(3) $\bar{a} \cdot m * (s + s') = am(s+s') = ams + ams' = \bar{a} \cdot m * s + \bar{a} \cdot m * s'$.
(4) $\bar{a}(\bar{a'} \cdot m * s)s' = \bar{a} \cdot (a'ms) * s' = a(a'ms)s' = (aa')m(ss') = (\bar{a}\bar{a'}) \cdot m * (ss')$.

Thus, it is proved that M is an $(R/I,S)$-module.

Proposition 3.6. *Let M be an (R,S)-module with $S^2 = S$ and A be ideal of R with $AMS = 0$. Then, M is a jointly second (R,S)-module if and only if M is jointly second $(R/A,S)$-modules.*

Proof. $(\Rightarrow)$. Since M is a jointly second (R,S)-module, $M \neq 0$. Let any ideal B of R with $A \subseteq B$. Since M is a jointly second (R,S)-module, either $BMS = 0$ or $BMS = M$. Moreover since $AMS = 0$ for any ideal A of R, we obtain $(B/A)MS = M$ or $(B/A)MS = 0$. Based on Proposition 3.3, it is proved that M is a jointly second $(R/A,S)$-module.

$(\Leftarrow)$. Since M is a jointly second $(R/A,S)$-module, $M \neq 0$. Let any ideal C of R. Since $AMS = 0$ for any ideal A of R, $A \subseteq (0 :_R M)$ so that we obtain $CMS = \Big((C+A)/A\Big)MS$. Since M is a jointly second $(R/A,S)$-module, we have either $\Big((C+A)/A\Big)MS = M$ or $\Big((C+A)/A\Big)MS = 0$. So, we have $CMS = M$ or $CMS = 0$. Based on Proposition 3.3, it is shown that M is a jointly second (R,S)-module. ∎

Before we give the next properties, the following we give a property about jointly prime (R,S)-submodule.

Proposition 3.7. *Let M be an (R,S)-module with $S^2 = S$ and P be jointly prime (R,S)-submodule of M. Then $(P :_R M)$ is a prime ideal of R.*

Proposition 3.8. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then, a proper (R,S)-submodule X of M is a jointly prime (R,S)-submodule if and only if for any non-zero (R,S)-submodule K/X of M/X satisfy $(X :_R K) = (X :_R M)$.*

Proposition 3.9. *Let R be a ring such that every prime ideal is maximal. An (R,S)-module M with $S^2 = S$ is a jointly prime (R,S)-module if and only if M is a jointly second (R,S)-module.*

Proof. $(\Rightarrow)$. Let M be a jointly prime (R,S)-module. It means that $(0 :_R M)$ is a prime ideal of R and $M \neq 0$. For any proper (R,S)-submodule N of M, we form (R,S)-module factor M/N. Clearly that $(0 :_R M) \subseteq (N :_R M)$. Since $(0 :_R M)$ is a prime ideal of R, $(0 :_R M)$ is a maximal ideal, so that we have $(0 :_R M) = (N :_R M)$. Thus, based on Proposition 3.2 we have M is a jointly second (R,S)-module.

$(\Leftarrow)$. Let M be a jointly second (R,S)-module. Then $M \neq 0$. And based on Proposition 3.4 we have $(0 :_R M)$ is a prime ideal of R. Let any non-zero (R,S)-submodule N of M. Since every prime ideal of R is maximal ideal, we obtain $(0 :_R N) \subseteq (0 :_R M)$. Furthermore, clearly that $(0 :_R M) \subseteq (0 :_R N)$. Hence, we have $(0 :_R M) = (0 :_R N)$. Hence M is a jointly prime (R,S)-module. ∎

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