HASIL CEK5_60140768

by 60140768 Pmat

Submission date: 19-May-2022 09:44AM (UTC+0700)
Submission ID: 1839516904
File name: 5 Pendidikan Matematika - 60140768 - JURNAL (SEAMS Bull -terbit).pdf (117.05K)
Word count: 4152
Character count: 16203

Southeast Asian Bulletin of Mathematics (2021) 45: 561–570

Southeast Asian
Bulletin of
Mathematics

On Jointly Second (R, S)-Submodules

Dian Ariesta Yuwaningsih
Department of Mathematics Education, Universitas Ahmad Dahlan, Indonesia
Email: dian.ariesta@pmat.uad.ac.id

Indah Emilia Wijayanti
Department of Mathematics, Universitas Gadjah Mada, Indonesia
Email: ind_wijayanti@ugm.ac.id

Received 26 August 2019
Accepted 26 December 2019

Communicated by Nguyen Van Sanh

AMS Mathematics Subject Classification(2000): 16D10, 16D80

Abstract. Let R and S be commutative rings and M be an (R, S)-module. In this paper, we present the dual notion of jointly prime (R, S)-submodules, that is called jointly second (R, S)-submodules, and we investigate some properties of them. We give a necessary and sufficient condition for an (R, S)-submodule being jointly second (R, S)-submodules. Moreover, we present the definition of jointly second (R, S)-modules and present a condition for jointly prime (R, S)-modules being jointly second (R, S)-modules and vice versa.

Keywords: Second submodules; Coprime submodules; Jointly prime; Second modules.

1. Preliminaries

Throughout this article, R and S will denote commutative rings and M be an additive Abelian group. Furthermore, $\mathbb{Z}$ will denote the ring of integers.

Let M be a left R-module. Based on [5], a proper submodule N of M is called prime if for each $r \in R$ and $m \in M$ such that $rm \in N$ implies either $m \in M$ or $rM \subseteq N$. This definition has been generalized by [9]. Based on [9], a proper submodule N of M is called prime if for each $r \in R$, the homomorphism $g_r : M/N \to M/N$ is either injective or zero. M is said to be second modules if the zero submodule of M is prime.

Moreover, [9] also introduced the dual notion of prime submodules, that is called second submodules. A non-zero submodule N of M is said to be second if for each $r \in R$, the homomorphism $f_r : N \to N$ is either surjective or zero. Further, [9] also provided some properties of second submodules. Several researchers have studied this second submodule, among of them are [1], [2], [3], [4], [8], and [6].

On the other hand, [7] defined the structure (R, S)-modules as a generalization of (R, S)-bimodules. Further, [7] also introduced the definition of jointly prime (R, S)-submodule, when R and S are arbitrary rings (not necessary commutative). A proper (R, S)-submodule P of M is called jointly prime if for any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. When R and S are commutative rings, we have a proper (R, S)-submodule P of M is called jointly prime if for any ideal I of R, ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. Moreover, the research about (R, S)-modules have been developed in [10] and [11].

The main purpose is to provide some information concerning the dual notion of jointly prime (R, S)-submodules. We will call this dual notion as jointly second (R, S)-submodules.

In Section 2, we give the definition of jointly second (R, S)-submodules and some examples of them. Moreover, we also provide some properties of jointly second (R, S)-submodules. Among of them are the necessary and sufficient condition for an (R, S)-submodule being jointly second; every simple (R, S)-submodule is jointly second; the annihilator of jointly second (R, S)-submodules is prime; and every jointly second (R, S)-submodule contained in maximal jointly second (R, S)-submodule.

In [9], a left R-module M is called second if M is second submodules for itself. In Section 3, we give the definition of jointly second (R, S)-modules. Moreover, we provide some properties of them. At the end, we present a condition for jointly prime (R, S)-modules being jointly second (R, S)-modules and vice versa.

2. Some Properties of Jointly Second (R, S)-Submodules

In this section we present the definition of jointly second (R, S)-submodule and some properties of them. However, it should be noted earlier the definition of jointly prime (R, S)-submodules, which was introduced by [7] as follow.

Definition 2.1. *Let R and S be arbitrary rings. A proper (R, S)-submodule P of M is called jointly prime if for any left ideal I of R, right ideal J of S, and (R, S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$.*

When R and S are commutative rings, we have the definition of jointly prime (R,S)-submodule as follows.

Definition 2.2. *A proper (R,S)-submodule P of M is called jointly prime (R,S)-submodule if for any ideal I of R, ideal J of S, and (R,S)-submodule N of M such that $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$.*

The definition of jointly second (R,S)-submodule is given as follows.

Definition 2.3. *An (R,S)-submodule N of M is called jointly second (R,S)-submodule if $N \neq 0$ and for each $r \in R$, the homomorphism (R,S)-module $f_r : N \longrightarrow N$ with definition $f_r(n) = rnS$ for each $n \in N$, is an epimorphism or zero homomorphism.*

Note that if f_r is an epimorphism, we have $Im(f_r) = N$. However, if f_r is a zero homomorphism then for each $n \in N$ satisfies $f_r(n) = 0$, so $rnS = 0$. Because it applies to every $n \in N$, then we have $rNS = 0$.

Next, we give an example of jointly second (R,S)-submodule.

Example 2.4. Consider the $(\mathbb{Z},\mathbb{Z})$-module $\mathbb{Z}_{12}$.

(1) The $(\mathbb{Z},\mathbb{Z})$-submodule $N = \{\bar{0},\bar{6}\}$ of $\mathbb{Z}_{12}$ is a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}_{12}$. For $m \in \mathbb{Z}$, we construct an $(\mathbb{Z},\mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : N &\longrightarrow N \\ \bar{n} &\longmapsto f_m(\bar{n}) = \overline{mn\mathbb{Z}} \quad ,\forall \bar{n} \in N. \end{aligned}$$

If $m \in 2\mathbb{Z}$, then we obtain $f_m(N) = \{\bar{0}\}$.
If $m \notin 2\mathbb{Z}$, then we have $f_m(N) = N$.
Thus, f_m is a zero homomorphism or an epimorphism. Hence, it is proved that N is a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}_{12}$.

(2) The $(\mathbb{Z},\mathbb{Z})$-submodule $K = \{\bar{0},\bar{4},\bar{8}\}$ of $\mathbb{Z}_{12}$ is a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}_{12}$. For $m \in \mathbb{Z}_{12}$, we construct an $(\mathbb{Z},\mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : K &\longrightarrow K \\ \bar{k} &\longmapsto f_m(\bar{k}) = \overline{mk\mathbb{Z}} \quad ,\forall \bar{k} \in N. \end{aligned}$$

If $m \in 3\mathbb{Z}$, then we have $f_m(K) = \{\bar{0}\}$.
If $m \notin 3\mathbb{Z}$, then we get $f_m(K) = K$.
Thus, f_m is a zero homomorphism or an epimorphism. Hence, it is proved that K is a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}_{12}$.

Now, we give an example of (R,S)-submodule which is not jointly second.

Example 2.5. Let $2\mathbb{Z}$ be an $(\mathbb{Z},\mathbb{Z})$-module. An $(\mathbb{Z},\mathbb{Z})$-submodule $4\mathbb{Z}$ of $2\mathbb{Z}$ is not a jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $2\mathbb{Z}$. For any element $m \in \mathbb{Z}$, we construct an $(\mathbb{Z},\mathbb{Z})$-module homomorphism f_m with:

$$\begin{aligned} f_m : 4\mathbb{Z} &\longrightarrow 4\mathbb{Z} \\ a &\longmapsto f_m(a) = ma\mathbb{Z} \quad , \forall a \in 4\mathbb{Z}. \end{aligned}$$

If $m = 0$, then we have $f_m(4\mathbb{Z}) = \{0\}$.
But, if $m \neq 0$ then not necessary $f_m(4\mathbb{Z}) = 4\mathbb{Z}$.
For the example, let any element $m = 2 \in \mathbb{Z}$. Then we have:

$$f_m(4\mathbb{Z}) = 2(4\mathbb{Z})\mathbb{Z} \subseteq 8\mathbb{Z} \subseteq 4\mathbb{Z}$$

but $f_m(4\mathbb{Z}) \neq 4\mathbb{Z}$. So, we get $4\mathbb{Z}$ is not jointly second $(\mathbb{Z},\mathbb{Z})$-submodule of $2\mathbb{Z}$.

According to [7], for each (R,S)-submodule N of M, let the set

$$(K :_R M) = \{r \in R \mid rMS \subseteq K\}.$$

In general $(K :_R M)$ is only an additive subgroup of R. But if we have the condition $S^2 = S$, clearly that $(K :_R M)$ is an ideal of R. We may also say that $(K :_R M)$ is the annihilator of quotient (R,S)-module M/K over the ring R.

Now, we present some properties of jointly second (R,S)-submodules.

Proposition 2.6. *Let M be an (R,S)-module with $S^2 = S$. An (R,S)-submodule N of M is a jointly second (R,S)-submodule if and only if $(0 :_R N) = (K :_R N)$ for each proper (R,S)-submodule K of N.*

Proof. $(\Rightarrow)$. Let N be jointly second (R,S)-submodules of M. Then, $N \neq 0$ and for each $r \in R$, the (R,S)-module homomorphisms

$$\begin{aligned} f_r : N &\longrightarrow N \\ n &\longmapsto f_r(n) = rnS, \quad \forall n \in N \end{aligned}$$

is an epimorphism or zero homomorphism. Let any $x \in (K :_R N)$. Then $xNS \subseteq K \subset N$. If the homomorphism f_x is an epimorphism then $f_X(N) = N$, so we get $xNS = N \subseteq K$. A contradiction with $K \subset N$. So, f_x is a zero homomorphism. Thus, we obtain $f_x(N) = 0$ or $xNS = 0$, so $x \in (0 :_R N)$. Thus, we obtain $(K :_R N) \subseteq (0 :_R N)$. Moreover, let any $y \in (0 :_R N)$. Then $yNS = 0$. Since K is an (R,S)-submodule of M, $yNS = 0 \subseteq K$. Thus, $y \in (K :_R N)$. Hence, we obtain $(0 :_R N) \subseteq (K :_R N)$. Thus, it is proved that $(0 :_R N) = (K :_R N)$.

$(\Leftarrow)$. Let any proper (R,S)-submodule K of N satisfy $(0 :_R N) = (K :_R N)$. For any $r \in R$, we construct an (R,S)-module homomorphism

$$\begin{aligned} f_r : N &\longrightarrow N \\ n &\longmapsto f_r(n) = rnS, \quad \forall n \in N. \end{aligned}$$

Assume that f_r is not an epimorphism. We will show that f_r is a zero homomorphism. Since f_r not an epimorphism, $Im(f_r) \neq N$, so that $rNS \neq N$. Suppose $Im(f_r) = K$. Then $rNS = K$, so that $r \in (K :_R N)$. Since $(0 :_R N) = (K :_R N)$, $r \in (0 :_R N)$, so that $rNS = 0$. Consequently, we get $f_r(N) = 0$. Thus, it is proved that f_r is a zero homomorphisms. Hence, N is a jointly second (R, S)-submodule of M. ■

An (R, S)-module M is said to be simple if the (R, S)-submodule of M is only zero submodule and M itself.

Proposition 2.7. *Let M be an (R, S)-module with $S^2 = S$ and N a simple (R, S)-submodule of M. Then, N is a jointly second (R, S)-submodule of M.*

Proof. It is known that N is a simple (R, S)-submodule of M, so N only contains (R, S)-submodule $\{0\}$ and N itself. Since $\{0\}$ is the only proper (R, S)-submodule of N, we can form (R, S)-module factor $N/\{0\} = N$. Based on Proposition 2.6, we have $(0 :_R N) = (\{0\} :_R N)$. Thus, N is a jointly second (R, S)-submodule of M. ■

Proposition 2.8. *Let M be an (R, S)-module with $S^2 = S$ and (R, S)-submodule N of M with $N \neq 0$. Then the following statements are equivalent:*

(1) *N is a jointly second (R, S)-submodule of M.*
(2) *For any ideal A of R, $ANS = 0$ or $ANS = N$.*
(3) *$ANS = N$ for any ideal A of R not contained in $(0 :_R N)$.*
(4) *$ANS = N$ for any ideal A of R properly containing $(0 :_R N)$.*

Proof. (1) ⇒ (2). It is known that N is a jointly second (R, S)-submodule, meaning it fulfills $(0 :_R N) = (K :_R N)$ for each proper (R, S)-submodule K of N. We will show that $ANS = 0$ or $ANS = N$. Let any ideal A of R and assume that $ANS \neq N$. We will show that $ANS = 0$. Since $ANS \neq N$, ANS is a proper (R, S)-submodule of N, so that we can form (R, S)-module factor N/ANS. Suppose that $B = (0 :_R N/ANS)$. Then, we have

$$BNS = \left\{ \sum_{i=1}^{k} b_i n_i S \mid b_i \in B,\ n_i \in N \right\}.$$

Since $B = (0 :_R N/ANS)$, $BNS \subseteq ANS$, so that for each $s \in S$ satisfy $bns = a'n's'$ where $a' \in A$, $b \in B$, $n, n' \in N$, and $s' \in S$. As a result, we obtain

$$BNS = \left\{ \sum_{j=1}^{l} a_j n_j S \mid a_j \in A,\ n_j \in N \right\} = ANS.$$

Since N is a jointly second, we obtain $B = (0 :_R N/ANS) = (0 :_R N)$, so that $BNS = 0$. Thus, it is proved that $ANS = 0$.

(2) $\Rightarrow$ (3). Let any ideal A of R with $A \nsubseteq (0 :_R N)$. Then, $ANS \neq 0$. Based on the hypothesis we get $ANS = N$.

(3) $\Rightarrow$ (4). Let any ideal A of R with $(0 :_R N) \subset A$. That means $A \nsubseteq (0 :_R N)$. Based on the hypothesis we have $ANS = N$.

(4) $\Rightarrow$ (1). Let any proper (R, S)-submodule K of N. Suppose that $X = (K :_R N)$. We Have $(0 :_R N) \subseteq X$ and $XNS \subseteq K \neq N$ (since $(0 :_R N)$ is not proper subset of X). Because the only one that satisfied $N \neq XNS$ and $XNS \subseteq K$ is $XNS = 0$, we have $X = (0 :_R N)$. So, $(0 :_R N) = (K :_R N)$ for each proper (R, S)-submodule K of N. Hence, it is proved that K is a jointly second (R, S)-submodule of M. ■

Proposition 2.9. *Let M be an (R, S)-module with $S^2 = S$ and N be jointly second (R, S)-submodule of M. Then, $(0 :_R N)$ is a prime ideal of R.*

Proof. Let any ideal I and J of R such that $IJ \subseteq (0 :_R N)$. We will show that either $I \subseteq (0 :_R N)$ or $J \subseteq (0 :_R N)$. Since $IJ \subseteq (0 :_R N)$, we have $IJNS = 0$. Since N is a jointly second (R, S)-submodule, we have either $JNS = 0$ or $JNS = N$. If $JNS = N$, then $INS = IJNSS \subseteq IJNS = 0$, so that $INS = 0$. From here, we get $I \subseteq (0 :_R N)$. Moreover, if $JNS \neq N$ then $JNS = 0$, so that $J \subseteq (0 :_R N)$. Thus, we have either $I \subseteq (0 :_R N)$ or $J \subseteq (0 :_R N)$. Hence, $(0 :_R N)$ is a prime ideal of R. ■

Proposition 2.10. *Let M be an (R, S)-module with $S^2 = S$ and $(N_i)_{i \in I}$ be a chain of jointly second (R, S)-submodule of M. Then $N = \bigcup_{i \in I} N_i$ is jointly second (R, S)-submodule of M.*

Proof. Since $(N_i)_{i \in I}$ is a chain of jointly second (R, S)-submodule of M, $N = \bigcup_{i \in I} N_i$ is a non-zero (R, S)-submodule of M. Suppose formed $P_i = (0 :_R N_i)$ for each $i \in I$. Let any $i, j \in I$. Then $N_i \subseteq N_j$ or $N_j \subseteq N_i$, so we get either $P_j \subseteq P_i$ or $P_i \subseteq P_j$. Moreover, let any ideal A of R with $ANS \neq 0$. We will show that $ANS = N$. Since $ANS \neq 0$, there exist $n \in N_k$, $a \in A$, and $k \in I$ such that $anS \neq 0$. Consequently, we have $AN_kS \neq 0$. Since N_k is a jointly second (R, S)-submodule, $AN_kS = N_k \subseteq ANS$. If $P_i \subseteq P_k$ and $AN_kS \neq 0$ then $A \nsubseteq P_k$. As a result, $A \nsubseteq P_i$ so we have $N_i = AN_iS \subseteq ANS$. If $P_k \subseteq P_i$, that means $N_i \subseteq N_k$. Since $N_k = AN_kS \subseteq ANS$, we have $N_i \subseteq ANS$. Thus, we have $N_i \subseteq ANS$ for each $i \in I$. Clearly that $ANS \subseteq N$, so $ANS = N$. Hence, $N = \bigcup_{i \in I} N_i$ is jointly second (R, S)-submodule of M. ■

Proposition 2.11. *Let M be a non-zero (R, S)-module with $S^2 = S$. Then every jointly second (R, S)-submodule of M contained in maximal jointly second (R, S)-submodule of M.*

Proof. Let N be jointly second (R, S)-submodule of M. We construct the set $\mathfrak{J} = \{P \mid P \text{ jointly second } (R, S)\text{-submodule of } M \text{ with } N \subseteq P\}$. It is obviously that $\mathfrak{J} \neq \emptyset$ since $N \in \mathfrak{J}$. By using Zorn's Lemma, we will show that $\mathfrak{J}$ has a maximal element. Equivalently showing that every non-empty chain $\mathfrak{C}$ of $\mathfrak{J}$ has an upper bound in $\mathfrak{J}$. Let any non-empty chain $\mathfrak{C} \in \mathfrak{J}$ and form the set $Q = \bigcup_{K \in \mathfrak{C}} K$. Based on Proposition 2.10, Q is also jointly second (R, S)-submodule of M. Since $N \subseteq Q$, $Q \in \mathfrak{J}$ and Q is an upper bound of $\mathfrak{C}$. Thus, it is proved that every non-empty chain $\mathfrak{J}$ has an upper bound in $\mathfrak{J}$. Therefore, based on Zorn's Lemma there exist a jointly second (R, S)-submodule $N^* \in \mathfrak{J}$ that maximal between all jointly second (R, S)-submodules of $\mathfrak{J}$. Thus, it is proved that every jointly second (R, S)-submodule N contained in maximal jointly second (R, S)-submodule N^* of M. ∎

3. Jointly Second (R, S)-Modules

In this section, we present the definition of jointly second (R, S)-module and their properties. The definition of jointly second (R, S)-modules is given below.

Definition 3.1. *An (R, S)-module M is called a jointly second (R, S)-module if M is a jointly second (R, S)-submodule for itself.*

Now, we give some properties of jointly second (R, S)-modules. These properties based on the properties of jointly second (R, S)-submodule which was presented in the previous section.

Proposition 3.2. *Let M be a non-zero (R, S)-module with $S^2 = S$. Then, M is a jointly second (R, S)-module if $(0 :_R M) = (N :_R M)$ for every proper (R, S)-submodule N of M.*

Proof. Obviously from Proposition 2.6. ∎

Proposition 3.3. *Let M be a non-zero (R, S)-module with $S^2 = S$. The following statements are equivalent:*

(1) *M is a jointly second (R, S)-module.*
(2) *For any ideal A of R, $AMS = 0$ or $AMS = M$.*
(3) *$AMS = M$ for any ideal A of R not contained in $(0 :_R M)$.*
(4) *$AMS = M$ for any ideal A of R properly containing $(0 :_R M)$.*

Proof. Clearly from Proposition 2.8. ∎

Proposition 3.4. *Let M be an (R, S)-module with $S^2 = S$ and N be jointly second (R, S)-submodule of M. Then, $(0 :_R M)$ is a prime ideal of R.*

Proof. Evidently from Proposition 2.9. ∎

Proposition 3.5. *Let M be a simple (R, S)-module with $S^2 = S$. Then, M is a jointly second (R, S)-module.*

Proof. Obviously from Proposition 2.7. ■

Before proceeding to the next properties of jointly second (R, S)-modules, the following is given one of the properties of (R, S)-modules. This properties will be used in proving the necessary and sufficient conditions of jointly second (R, S)-module.

Let M be an (R, S)-module M and I an ideal of R that satisfy $I \subseteq Ann_R(M)$. We defined the scalar multiplication operation:

$$\_ \cdot \_ * \_ : R/I \times M \times S \longrightarrow M$$
$$(\bar{a}, m, s) \longrightarrow \bar{a} \cdot m * s := ams$$

for each $\bar{a} \in R/I$, $m \in M$, and $s \in S$. Clearly that this scalar multiplication operation is closed. Moreover, we can show that this scalar multiplication operation is well-defined. Let any $\bar{a}, \bar{a'} \in R/I$, $m, m' \in M$, and $s, s' \in S$ with $(\bar{a}, m, s) = (\bar{a'}, m', s')$. This means that $\bar{a} = \bar{a'}$, $m = m'$, and $s = s'$. Since $\bar{a} = \bar{a'}$, we have $a - a' \in I$. Since $I \subseteq Ann_R(M)$, $(a - a')ms = 0$ so $ams = a'ms = a'm's'$. Thus, we have $\bar{a} \cdot m * s = \bar{a'} \cdot m' * s'$. Hence, it is proved that this scalar multiplication operation is well-defined.

Furthermore, we will show that an (R, S)-module M is an $(R/I, S)$-module over the scalar multiplication operation which is defined above. Let any $\bar{a}, \bar{a'} \in R/I$, $m, n \in M$, and $s, s' \in S$. Then we have:

(1) $\bar{a} \cdot (m + n) * s = a(m + n)s = ams + ans = \bar{a} \cdot m * s + \bar{a} \cdot n * s$.
(2) $(\bar{a} + \bar{a'}) \cdot m * s = (\overline{a + a'}) \cdot m * s = (a + a')ms = ams + a'ms = \bar{a} \cdot m * s + \bar{a'} \cdot m * s$.
(3) $\bar{a} \cdot m * (s + s') = am(s + s') = ams + ams' = \bar{a} \cdot m * s + \bar{a} \cdot m * s'$.
(4) $\bar{a}(\bar{a'} \cdot m * s)s' = \bar{a} \cdot (a'ms) * s' = a(a'ms)s' = (aa')m(ss') = (\bar{a}\bar{a'}) \cdot m * (ss')$.

Thus, it is proved that M is an $(R/I, S)$-module.

Proposition 3.6. *Let M be an (R, S)-module with $S^2 = S$ and A be ideal of R with $AMS = 0$. Then, M is a jointly second (R, S)-module if and only if M is jointly second $(R/A, S)$-modules.*

Proof. $(\Rightarrow)$. Since M is a jointly second (R, S)-module, $M \neq 0$. Let any ideal B of R with $A \subseteq B$. Since M is a jointly second (R, S)-module, either $BMS = 0$ or $BMS = M$. Moreover since $AMS = 0$ for any ideal A of R, we obtain $(B/A)MS = M$ or $(B/A)MS = 0$. Based on Proposition 3.3, it is proved that M is a jointly second $(R/A, S)$-module.

$(\Leftarrow)$. Since M is a jointly second $(R/A, S)$-module, $M \neq 0$. Let any ideal C of R. Since $AMS = 0$ for any ideal A of R, $A \subseteq (0 :_R M)$ so that we obtain $CMS = \big((C + A)/A\big)MS$. Since M is a jointly second $(R/A, S)$-module, we have either $\big((C + A)/A\big)MS = M$ or $\big((C + A)/A\big)MS = 0$. So, we have $CMS = M$ or $CMS = 0$. Based on Proposition 3.3, it is shown that M is a jointly second (R, S)-module. ■

Before we give the next properties, the following we give a property about jointly prime (R, S)-submodule.

Proposition 3.7. *Let M be an (R, S)-module with $S^2 = S$ and P be jointly prime (R, S)-submodule of M. Then $(P :_R M)$ is a prime ideal of R.*

Proposition 3.8. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then, a proper (R, S)-submodule X of M is a jointly prime (R, S)-submodule if and only if for any non-zero (R, S)-submodule K/X of M/X satisfy $(X :_R K) = (X :_R M)$.*

Proposition 3.9. *Let R be a ring such that every prime ideal is maximal. An (R, S)-module M with $S^2 = S$ is a jointly prime (R, S)-module if and only if M is a jointly second (R, S)-module.*

Proof. $(\Rightarrow)$. Let M be a jointly prime (R, S)-module. It means that $(0 :_R M)$ is a prime ideal of R and $M \neq 0$. For any proper (R, S)-submodule N of M, we form (R, S)-module factor M/N. Clearly that $(0 :_R M) \subseteq (N :_R M)$. Since $(0 :_R M)$ is a prime ideal of R, $(0 :_R M)$ is a maximal ideal, so that we have $(0 :_R M) = (N :_R M)$. Thus, based on Proposition 3.2 we have M is a jointly second (R, S)-module.

$(\Leftarrow)$. Let M be a jointly second (R, S)-module. Then $M \neq 0$. And based on Proposition 3.4 we have $(0 :_R M)$ is a prime ideal of R. Let any non-zero (R, S)-submodule N of M. Since every prime ideal of R is maximal ideal, we obtain $(0 :_R N) \subseteq (0 :_R M)$. Furthermore, clearly that $(0 :_R M) \subseteq (0 :_R N)$. Hence, we have $(0 :_R M) = (0 :_R N)$. Hence M is a jointly prime (R, S)-module. ■

References

[1] H. Ansari-Toroghy and F. Farshadifar, On the dual notion of prime submodules, *Algebra Colloqium* **19** (1) (2012) 1109–1116.

[2] H. Ansari-Toroghy and F. Farshadifar, On the dual notion of prime submodules II, *Mediteranian Journal of Mathematics* **9** (2012) 327–336.

[3] S. Çeken, M. Alkan, P.F. Smith, Second modules over noncommutative rings, *Communications in Algebra* **41** (1) (2013) 83–98.

[4] S. Çeken, M. Alkan, P.F. Smith, The dual notion of the prime radical of a module, *Journal of Algebra* **329** (2013) 265–275.

[5] J. Dauns, Prime modules, *Journal für die reine und angewandte Mathematik* **298** (1978) 156–181.

[6] R.I. Khalaf, Dual notion of prime modules, *IBN AL-HAITAM Journal for Pure and Application Sciences* 23 (3) (2012).

[7] T. Khumprapussorn, S. Pianskool, M. Hall, (R, S)-modules and their fully and jointly prime submodules, *International Mathematical Forum* **7** (33) (2012) 1631–1643.

[8] I.E. Wijayanti, D.A. Yuwaningsih, S. Wahyuni, On weakly second submodules over noncommutative rings, *Southeast Asian Bull. Math.* **42** (2018) 781–790.

[9] S. Yassemi, The dual notion of prime submodules, *Arc. Math. Brno* **37** (2001) 273–278.

[10] D.A. Yuwaningsih and I.E. Wijayanti, On jointly prime radicals of (R,S)-modules, *Journal of the Indonesian Mathematical Society* **21** (1) (2015) 25-34.

[11] D.A. Yuwaningsih, I.E. Wijayanti, P.W. Prasetyo, On (R,S)-modules homomorphisms, *IOP Conf. Series: Journal of Physics: Conf. Series* **1188** (2019), 012114, 7 pages.

HASIL CEK5_60140768

ORIGINALITY REPORT

13%	15%	7%	6%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	www.coursehero.com Internet Source	5%
2	www.seams-bull-math.ynu.edu.cn Internet Source	5%
3	S. Çeken, M. Alkan, P. F. Smith. "Second Modules Over Noncommutative Rings", Communications in Algebra, 2013 Publication	2%

Exclude quotes	On	Exclude matches	< 2%
Exclude bibliography	On		