HASIL CEK3_60140768

by 60140768 Pmat

Submission date: 19-May-2022 09:44AM (UTC+0700)
Submission ID: 1839516859
File name: 3 Pendidikan Matematika - 60140768 - JURNAL (JIMS-terbit).pdf (349.41K)
Word count: 4018
Character count: 15660

J. Indones. Math. Soc.
Vol. 26, No. 02 (2020), pp. 234–241.

SOME PROPERTIES OF LEFT WEAKLY JOINTLY PRIME (R,S)-SUBMODULES

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Abstract. Let R and S be commutative rings and M be an (R,S)-module. A proper (R,S)-submodule P of M is called a left weakly jointly prime if for each elements a and b in R and (R,S)-submodule K of M with $abKS \subseteq P$ implies either $aKS \subseteq P$ or $bKS \subseteq P$. In this paper, we present some properties of left weakly jointly prime (R,S)-submodule. We give some necessary and sufficient conditions of left weakly jointly prime (R,S)-submodules. Moreover, we present that every left weakly jointly prime (R,S)-submodule contains a minimal left weakly jointly prime (R,S)-submodule. At the end of this paper, we show that in left multiplication (R,S)-module, every left weakly jointly prime (R,S)-submodule is equal to jointly prime (R,S)-submodules.

Keywords and Phrases: weakly prime submodule, jointly prime submodule, prime submodule

1. INTRODUCTION

Throughout this paper, ring R and ring S will denote commutative rings, and R-module M means an Abelian group under addition. The concept of the R-module has been studied in depth in Adkins [1].

An R-module has been generalized into an (R,S)-bimodule. When R and S are arbitrary rings, Khumprapussorn *et al.* [7] have generalized (R,S)-bimodule into (R,S)-module. An (R,S)-module has an (R,S)-bimodule structure when both rings R and S have central idempotent elements. When R and S are rings with identity, we have an (R,S)-module is also an (R,S)-bimodule.

Moreover, an (R,S)-submodule of an (R,S)-module M is a subgroup N of M such that $rns \in N$ for all $r \in R$, $n \in N$, and $s \in S$. Let P be a proper (R,S)-submodule of M. By Khumprapussorn *et al.* [7], a proper (R,S)-submodule P of M is called jointly prime if for each left ideal I of R, right ideal J of S, and

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2020 Mathematics Subject Classification: 16D10, 16D80.
Received: 19-05-2019, accepted: 17-11-2019.

(R, S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. If R and S are commutative rings, we have a proper (R, S)-submodule P of M is called jointly prime if for each ideal I of R, ideal J of S, and (R, S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. The concept of jointly prime (R, S)-submodules when R and S are arbitrary rings have been studied by Khumprapussorn *et al.*[7] and continued by Yuwaningsih and Wijayanti [8].

On module theory, a proper submodule N of an R-module M is called prime if for each element a of R and element $m \in M$ with $am \in N$ implies $m \in N$ or $aM \subseteq N$. Prime submodules have been introduced and studied by Dauns [6]. As time went by, the researchers began to generalize the definition of prime submodules to weakly prime submodules. A proper submodule N of M is called weakly prime if for each submodule P of M and elements a, b of R satisfy $abP \subseteq N$, implies either $aP \subseteq N$ or $bP \subseteq N$. Weakly prime submodules have been introduced by Behboodi and Koohy [4]. Moreover, the studied about weakly prime submodules have been continued by Azizi [2], Behboodi [5], and Azizi [3].

In this paper, we present some properties of left weakly jointly prime (R, S)-submodules as the generalization of jointly prime (R, S)-submodules. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and elements a, b of R such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. Moreover, we present some properties of left weakly jointly prime (R, S)-submodules. Some of these properties are as follows: a proper (R, S)-submodules is left weakly jointly prime if and only if the annihilator of this quotient (R, S)-module over ring R is a prime ideal of R; when we give a left weakly jointly prime (R, S)-submodule, the set of that annihilator over ring R form a chain of prime ideals; if a left weakly jointly prime (R, S)-submodule is irreducible then it is a jointly prime; any left weakly jointly prime (R, S)-submodule P of M contains a minimal left weakly jointly prime (R, S)-submodule; and every left weakly jointly prime (R, S)-submodule is equal to jointly prime (R, S)-submodule in left multiplication (R, S)-module.

2. LEFT WEAKLY JOINTLY PRIME (R, S)-SUBMODULES

Before we present the definition of left weakly jointly prime (R, S)-submodules, we describe first the jointly prime (R, S)-submodule. As we have already stated earlier, when R and S are commutative rings, a proper (R, S)-submodule P of M is called jointly prime if for each ideal I of R, ideal J of S, and (R, S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. When R and S are arbitrary rings, Khumprapussorn *et al.* [7] have given some characteristic of jointly prime (R, S)-submodules. In this paper, we modify those characteristics when R and S are commutative rings as follows.

Proposition 2.1. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. Then the following statements are equivalent.*

(1) P is jointly prime.

(2) *For all ideal I of R, $m \in M$, and ideal J of S, $ImJ \subseteq P$ implies $IMJ \subseteq P$ or $m \in P$.*
(3) *For all $a \in R$, $m \in M$, and $b \in S$, $amb \in P$ implies $aMb \subseteq P$ or $m \in P$.*
(4) *For all $a \in R$ and $m \in M$, $amS \subseteq P$ implies $aMS \subseteq P$ or $m \in P$.*

A proper submodule P of R-module M is called weakly prime if for each submodule K of M and elements a, b of R such that $abK \subseteq P$ implies either $aK \subseteq P$ or $bK \subseteq P$. This definition has studied by Azizi [2]. Based on that definition, we present the definition of left weakly jointly prime (R, S)-submodules as follows.

Definition 2.2. *Let M be an (R, S)-module. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and elements $a, b \in R$ such that $abNS \subseteq P$ implies $aNS \subseteq P$ or $bNS \subseteq P$.*

Note that right weakly jointly prime (R, S)-submodules can be defined and studied analogously. Now, if we have a condition $a \in RaS$ for all $a \in M$, then we give the definition of left weakly jointly prime (R, S)-submodules as follows.

Definition 2.3. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and ideals I and J of R such that $IJNS \subseteq P$ implies either $INS \subseteq P$ or $JNS \subseteq P$.*

We can easily show that the two definitions of left weakly jointly prime (R, S)-submodule above are equivalent. Moreover, we give some example of left weakly jointly prime (R, S)-submodules as follows.

Example 2.4. *Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-module and $2\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. We can show that $2\mathbb{Z}$ is a left weakly jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule. Let any $a, b \in 2\mathbb{Z}$ with $a = 2k$ and $b = 2l$ and let any $(2\mathbb{Z}, 2\mathbb{Z})$-submodule $N = x\mathbb{Z}$ of $\mathbb{Z}$, for some $k, l, x \in \mathbb{Z}$. Clearly that $abN(2\mathbb{Z}) = (8klx)\mathbb{Z}^2 \subseteq (8klx)\mathbb{Z} \subseteq 2\mathbb{Z}$. We obtain $aN(2\mathbb{Z}) \subseteq 4kx\mathbb{Z} \subseteq 2\mathbb{Z}$ or $bN(2\mathbb{Z}) \subseteq 4lx\mathbb{Z} \subseteq 2\mathbb{Z}$. Thus, $2\mathbb{Z}$ is a left weakly jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$.*

Example 2.5. *Let R and S are commutative rings with*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{Z} \right\} \text{ and } S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in 2\mathbb{Z} \right\}.$$

Let an (R, S)-module M with

$$M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in 2\mathbb{Z} \right\}.$$

Easily we can check that an (R, S)-submodule X of M with

$$X = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in 2\mathbb{Z} \right\}$$

is a left weakly jointly prime (R, S)-submodule of M.

3. SOME PROPERTIES OF LEFT WEAKLY JOINTLY PRIME (R,S)-SUBMODULES

In this section, we present some properties of left weakly jointly prime (R,S)-submodules. However, before we recall the definition of annihilator of the quotient (R,S)-modules over a ring R as follows.

Definition 3.1. *Let M be an (R,S)-module and N be an (R,S)-submodule of M. We define the annihilator of quotient (R,S)-module M/N over a ring R is the set $(N :_R M) = \{r \in R \mid rMS \subseteq N\}$.*

In general $(N :_R M)$ is only an additive subgroup of R. But if we have the condition $S^2 = S$, clearly that $(N :_R M)$ is an ideal of R.

Now, we give some properties of left weakly jointly prime (R,S)-submodules. In the first property, we show that every jointly prime (R,S)-submodules is a left weakly jointly prime.

Proposition 3.2. *Let M be an (R,S)-module with $a \in RaS$ for all $a \in M$ and P is a jointly prime (R,S)-submodule of M. Then, P is a left weakly jointly prime (R,S)-submodule of M.*

Proof. Let any ideal I,J of R and an (R,S)-submodule N of M such that $IJNS \subseteq P$. Since R is commutative, then $I(RJNS)S = R(IJNS)S \subseteq IJNS \subseteq P$. Since P is jointly prime then we get $IMS \subseteq P$ or $RJNS \subseteq P$. Since $a \in RaS$ for all $a \in M$, we have $IMS \subseteq P$ or $JNS \subseteq P$. Hence, $INS \subseteq IMS \subseteq P$ or $JNS \subseteq P$. Hence, P is a left weakly jointly prime (R,S)-submodule of M. □

Now, we present some necessary and sufficient conditions of a proper (R,S)-submodule being left weakly jointly prime.

Proposition 3.3. *Let M be an (R,S)-module with $S^2 = S$ and N be a proper (R,S)-submodule of M. Then, N is a left weakly jointly prime (R,S)-submodule of M if and only if for each (R,S)-submodule K of M with $K \nsubseteq N$ satisfy $(N :_R K)$ is a prime ideal of R.*

Proof. ($\Rightarrow$). Let an (R,S)-submodule K of M with $N \subset K$ and ideal I and ideal J of R such that $IJ \subseteq (N :_R K)$. This means $IJKS \subseteq N$. Since N is a left weakly jointly prime (R,S)-submodule of M, then $IKS \subseteq N$ or $JKS \subseteq N$. Consequently, we obtain $I \subseteq (N :_R K)$ or $J \subseteq (N_R K)$. Hence, $(N :_R K)$ is a prime ideal of R.
($\Leftarrow$). Let an ideal I and J of R and (R,S)-submodule L of M such that $IJLS \subseteq N$. Then, $IJ(L+N)S \subseteq N$, so $IJ \subseteq (N :_R L+N)$. Since $(N :_R L+N)$ is prime ideal, then we obtain $I \subseteq (N :_R L+N)$ or $J \subseteq (N :_R L+N)$. Consequently, we obtain $I(L+N)S \subseteq N$ or $J(L+N)S \subseteq N$. Thus, $ILS \subseteq N$ or $JLS \subseteq N$. Hence, N is a left weakly jointly prime (R,S)-submodule of M. □

Proposition 3.4. *Let M be an (R,S)-module with $S^2 = S$ and N be a proper (R,S)-submodule of M. Then, N is a left weakly jointly prime (R,S)-submodule if and only if for every $a,b \in R$ and $m \in M \setminus N$ satisfy $abmS \subseteq N$ implies $amS \subseteq N$ or $bmS \subseteq N$. On the other word, $(N :_R m)$ is a prime ideal of R.*

PROOF. ($\Rightarrow$). Let any $a, b \in R$ and $m \in M \setminus N$. We construct $\langle m \rangle$ is an (R, S)-submodule of M. Then we get

$$abmS \subseteq ab\langle m \rangle S \subseteq N.$$

Since N is a left weakly jointly prime (R, S)-submodule of M, then we have $amS \subseteq a\langle m \rangle S \subseteq N$ or $bmS \subseteq b\langle m \rangle S \subseteq N$.
($\Leftarrow$). Let any $a, b \in R$ and (R, S)-submodule K of M with $K \nsubseteq N$ such that $abKS \subseteq N$. For each $k \in K$, we have $abkS \subseteq N$. Based on the hypothesis, $akS \subseteq N$ or $bkS \subseteq N$. Thus, $aKS \subseteq N$ or $bKS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule of M. □

Proposition 3.5. *Let M be an (R, S)-module with $S^2 = S$ and N is a proper (R, S)-submodule of M. If M satisfies $a \in RaS$ for all $a \in M$, then the following statements are equivalent.*

(1) N is a left weakly jointly prime (R, S)-submodule of M.
(2) For any $x, y \in M$, if $(N :_R x) \neq (N :_R y)$, then $N = (N + RxS) \cap (N + RyS)$.

PROOF. ($1 \Rightarrow 2$). Let $r \in (N :_R x) \setminus (N :_R y)$, where $r \in R$ i.e. $rxS \subseteq N$ and $ryS \nsubseteq N$. Since by Proposition 3.4, $(N :_R y)$ is a prime ideal of R, it's easy to see that $(N :_R y) = (N :_R ryS)$. If we let $t \in (N + RxS) \cap (N + RyS)$, then $t = n_1 + r_1xs_1 = n_2 + r_2ys_2$, where $n_1, n_2 \in N$, $r_1, r_2 \in R$, and $s_1, s_2 \in S$. Note that

$$rtS = rn_1S + rr_1xs_1S = rn_2S + rr_2ys_2S$$

so $rtS = rn_1S + r_1(rx)s_1S = rn_2S + r_2(ry)s_2S$, where $rn_1S, rn_2S, r_1rxS \subseteq N$, so we obtain $r_2(ry)S \subseteq N$. Since $S^2 = S$, then $r_2(ry)SS \subseteq N$ so we get $r_2 \in (N :_R ryS) = (N :_R y)$. Then, $r_2yS \subseteq N$ so $t = n_2 + r_2ys_2 \in N$. Hence, it's proved that $(N + RxS) \cap (N + RyS) \subseteq N$. Furthermore, it's clear that $N \subseteq (N + RxS) \cap (N + RyS)$. Thus, $N = (N + RxS) \cap (N + RyS)$.
($2 \Rightarrow 1$). Let $r_1, r_2 \in R$ and $a \in M$ where $r_1r_2aS \subseteq N$. Since $S^2 = S$, then $r_1r_2aSS \subseteq N$. If $r_1aS \nsubseteq N$, we will show that $r_2aS \subseteq N$. From $r_1r_2aSS \subseteq N$, we obtain $r_1 \in (N :_R r_2aS) \setminus (N :_R a)$. Consequently, $(N :_R r_2aS) \neq (N :_R a)$. Put $x = r_2aS$ and $y = a$, then by our assumption we get $N = (N+Rr_2aS^2) \cap (N+RaS)$. Since $r_2aS \subseteq Rr_2aS^2 \subseteq N + Rr_2aS$ and $r_2aS \subseteq RaS \subseteq N + RaS$, we obtain $r_2aS \subseteq N$. □

Before we present the next properties, we recall the definition of an irreducible (R, S)-submodule as follow.

Definition 3.6. *An (R, S)-submodule N of M is called irreducibe if for each (R, S)-submodule N_1 and N_2 of M such that $N = N_1 \cap N_2$ implies either $N = N_1$ or $N = N_2$.*

Proposition 3.7. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$, N be a left weakly jointly prime (R, S)-submodule of M, and $x, y \in M$.*

(1) If $rxs \in N$ where $r \in R$, $s \in S$, then $N = (N + RxS) \cap (N + RryS)$.

(2) If N is an irreducible (R,S)-submodule, then N is a jointly prime (R,S)-submodule of M.

PROOF.

(1) If $rys \in N$, then clearly that $N = (N + RxS) \cap (N + RrysS)$. Since $S^2 = S$, then $N = (N + RxS) \cap (N + RryS)$. Moreover, if $rys \notin N$, then $(N :_R x) \neq (N :_R y)$. By Proposition 3.5, then $N = (N+RxS)\cap(N+RyS)$, so we have

$$N \subseteq (N + RxS) \cap (N + RryS) \subseteq (N + RxS) \cap (N + RyS) = N.$$

Thus, $N = (N + RxS) \cap (N + RryS)$.

(2) Let any $r \in R$, $s \in S$, and $x \in M$ such that $rxs \in N$. By part (a), for each $y \in M$ we have $N = (N + RxS) \cap (N + RryS)$. Since N is irreducible then $N = N + RxS$ or $N = N + RryS = N + NRrrysS$. We have $x \in N$ or $rys \in N$, so $x \in N$ or $rMs \subseteq N$. Thus, by Proposition 2.1 we have N is a jointly prime (R,S)-submodule of M. □

Let M be an (R,S)-module and N be an (R,S)-submodule of M. For every $a \in R$ we consider $(N :_M a)$ to be:

$$(N :_M a) = \{m \in M \mid amS \subseteq N\}.$$

We can show that $(N :_M a)$ is an (R,S)-submodule of M containing N.

Proposition 3.8. *Let M be an (R,S)-module with $S^2 = S$ and N be a left weakly jointly prime (R,S)-submodule of M. Then, for every $a, b \in R$ satisfy*

$$(N :_M ab) = (N :_M a) \cup (N :_M b).$$

PROOF. Let any $x \in (N :_M ab)$, then $abxS \subseteq N$. Since N is a left weakly jointly prime (R,S)-submodule, then $axS \subseteq N$ or $bxS \subseteq N$. So, we get $x \in (N :_M a)$ or $x \in (N :_M b)$, then $(N :_M ab) \subseteq (N :_M a) \cup (N :_M b)$. Next, let any $y \in (N :_M a) \cup (N :_M b)$. Then, $ayS \subseteq N$ or $byS \subseteq N$. Since $S^2 = S$ and N is an (R,S)-submodule of M, then we get

$$abyS = a(byS)S \subseteq aNS \subseteq N.$$

Hence, we obtain $y \in (N :_M ab)$. Thus, $(N :_M a) \cup (N :_M b) \subseteq (N :_M ab)$. Therefore, it has shown that $(N :_M ab) = (N :_M a) \cup (N :_M b)$. □

The following lemma gives us a property about the necessary and sufficient conditions of left weakly jointly prime (R,S)-submodules.

Lemma 3.9. *Let M be an (R,S)-module with $S^2 = S$ and N be a proper (R,S)-submodule of M. Then, N is a left weakly jointly prime (R,S)-submodule if and only if for each $a, b \in R$ satisfy $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.*

PROOF. $(\Rightarrow)$. Let N is a left weakly jointly prime (R,S)-submodule of M. Since $(N :_M ab) = (N :_M a) \cup (N :_M b)$ is an (R,S)-submodule of M, then $(N :_M a) \subseteq (N :_M b)$ or $(N :_M b) \subseteq (N :_M a)$. Therefore, we have $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.

($\Leftarrow$). Let any $a, b \in R$ and $m \in M$ with $abmS \subseteq N$. So we get $m \in (N :_M ab)$. Based on the hypothesis, $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. So $m \in (N :_M a)$ or $m \in (N :_M b)$. Consequently, $amS \subseteq N$ or $bmS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule of M. $\square$

A weakly jointly prime (R, S)-submodule P of M is called minimal if it is minimal in the class of weakly jointly prime (R, S)-submodules of M. In the next proposition, we present that any left weakly jointly prime (R, S)-submodule contains a minimal left weakly jointly prime (R, S)-submodule.

Proposition 3.10. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. Any left weakly jointly prime (R, S)-submodule P of M contains a minimal left weakly jointly prime (R, S)-submodule.*

PROOF. Let any weakly jointly prime (R, S)-submodule P of M and let $\mathfrak{J}$ be the set of all weakly jointly prime (R, S)-submodules of M that contained in P. Clearly, $\mathfrak{J} \neq \emptyset$ since $P \in \mathfrak{J}$. By using Zorn's Lemma, we will show that $\mathfrak{J}$ contains a minimal element. Equivalently, we show that every nonempty chain in $\mathfrak{J}$ has a lower bound in $\mathfrak{J}$. Let any nonempty chain $\mathfrak{G} \subseteq \mathfrak{J}$. We can construct the set $Q = \bigcap_{K \in \mathfrak{G}} K$. Then, clearly Q is an (R, S)-submodule of M and $Q \subseteq P$. We claim that Q is a weakly jointly prime (R, S)-submodule of M. Let any ideal I, J of R and an (R, S)-submodule N of M such that $IJNS \subseteq Q$ but $JNS \nsubseteq Q$. We will show that $INS \subseteq Q$. Let any element $n \in JNS \setminus Q$. Then, there exist $K' \in \mathfrak{G}$ such that $n \notin K'$. Since K' is a left weakly jointly prime (R, S)-submodule of M, then from $IJNS \subseteq Q \subseteq K'$ implies $INS \subseteq K'$. Moreover, let any $L \in \mathfrak{G}$. Since $\mathfrak{G}$ is a chain of $\mathfrak{J}$, then $K' \subseteq L$ or $L \subseteq K'$. If $K' \subseteq L$, then we obtain $INS \subseteq K' \subseteq L$. If $L \subseteq K'$, then we get $n \notin L$. Since L is a left weakly jointly prime (R, S)-submodule of M, then from $IJNS \subseteq Q \subseteq L$ implies $INS \subseteq L$. Thus, we obtain $INS \subseteq L$ for any $L \in \mathfrak{G}$ and so $INS \subseteq Q$. Hence, proved that Q is a left weakly jointly prime (R, S)-submodule of M. Since $Q \subseteq P$, then $Q \in \mathfrak{J}$ and Q is a lower bound of $\mathfrak{G}$. Thus, it's proved that every nonempty chain of $\mathfrak{J}$ has a lower bound in $\mathfrak{J}$. Based on Zorn's Lemma, there exist a left weakly jointly prime (R, S)-submodule $P^* \in \mathfrak{J}$ that minimal among the left weakly jointly prime (R, S)-submodules in $\mathfrak{J}$. Thus, it's proved that any left weakly jointly prime (R, S)-submodules P contain minimal left weakly jointly prime (R, S)-submodule P^* of M. $\square$

From Khumprapussorn *et al.*[7], we know that an (R, S)-module M is called left multiplication (R, S)-module provided that for each (R, S)-submodule N of M there exists an ideal I of R such that $N = IMS$. We have the characterization of jointly prime (R, S)-submodule of left multiplication (R, S)-modules as follow.

Theorem 3.11. *Let M be a left multiplication (R, S)-module with $S^2 = S$. Then, P is a jointly prime (R, S)-submodule of M if and only if $(P :_R M)$ is a prime ideal of R.*

In the following proposition, we present that every left weakly jointly prime (R, S)-submodules is equal to jointly prime (R, S)-submodules in left multiplication (R, S)-modules.

Proposition 3.12. *Let M be a left multiplication (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. An (R, S)-submodule N of M is jointly prime if and only if N is left weakly jointly prime (R, S)-submodules of M.*

Proof. $(\Rightarrow)$. it's obvious.
$(\Leftarrow)$. Let N is a left weakly jointly prime (R, S)-submodule of M. Then $(N :_R M)$ is a prime ideal of R. Since M is a left multiplication (R, S)-module and ring S satisfy $S^2 = S$, then based on Theorem 3.11 we have N is a jointly prime (R, S)-submodule of M. □

4. CONCLUDING REMARKS

Further work on the properties of left weakly jointly prime (R, S)-submodules can be carried out. For example, the investigation of properties of left weakly jointly prime radicals of an (R, S)-module.

Acknowledgment. The authors thank to the referees for valuable comments and suggestions. This work is supported by Grant Penelitian Dosen Pemula, Ministry of Research, Technology and Higher Education of the Republic of Indonesia, 2018.

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