HASIL CEK1_60140768

by 60140768 Pmat

Submission date: 19-May-2022 09:44AM (UTC+0700)
Submission ID: 1839516831
File name: 1 Pendidikan Matematika - 60140768 - JURNAL (AEJM-terbit).pdf (123.73K)
Word count: 3003
Character count: 12207

Asian-European Journal of Mathematics
Vol. 12, No. 5 (2019) 1950072 (7 pages)

DOI: 10.1142/S1793557119500724

$S(0)$-Weakly second submodules

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Communicated by A. R. Rajan
Received February 15, 2018
Accepted April 5, 2018
Published May 25, 2018

We introduce the dual notions of $S(N)$-weakly prime submodules, that is, $S(0)$-weakly second submodules in a commutative ring with identity. We investigate the properties of $S(0)$-weakly second submodules and obtain some related useful characterizations of this dualization. Moreover, we also prove some properties of $S(0)$-weakly second submodules related to multiplication and comultiplication modules.

Keywords: Second submodules; weakly prime submodules; weakly second submodules.

AMS Subject Classification: 16D10, 16D80

1. Preliminaries

Dauns introduced the notion of prime modules in his paper [6]. This primeness has been generalized into weakly prime module in [3, 2, 1]. If M is an R-module, P is a maximal ideal of R, then we obtain a multiplicative closed set $T = R\backslash P$. Moreover, by these situations, we can construct the ring of fraction of R denoted by R_P and the module of fraction M_P over R_P. For any submodule N of M, we can extend it to a submodule in M_P and denote it by N_P. Moreover, in his paper, Jabbar [7] showed the relationship between the primeness of module M and its

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module of fraction M_P, where P is a maximal ideal of R. Naturally, any prime submodule N of M implies N_P is also a prime submodule of M_P, but the converse is not always true. Yassemi in [8] has defined second submodules as the dual of prime submodules. Furthermore, second submodules have been intensively studied by Çeken *et al.* in [4, 5]. Some authors also call the second submodules as coprime submodules.

Assume that R is a commutative ring. Then, a proper submodule N of M is said to be weakly prime if for any $r, s \in R$ and submodule K of M with $rsK \subseteq N$ implies either $rK \subseteq N$ or $sK \subseteq N$. Equivalently, a proper submodule N of M is said to be a weakly prime submodule if for any submodule K of M, where $N \subsetneq K$, $\mathrm{Ann}_R(K/N)$ is a prime ideal. Moreover, Ansari and Toroghy [1] introduced the dual notion of weakly prime submodules over a commutative ring with identity, that is weakly second submodules. A nonzero submodule N of M is called a weakly second submodule if for each $r, s \in R$ and submodule K of M such that $rsN \subseteq K$, implies either $rN \subseteq K$ or $sN \subseteq K$.

The relationship between the weakly secondness and its module of fraction is interesting to be considered. The combination of the ideas of Jabbar [7] and Yassemi [8] obtains a dualization of $S(N)$-locally prime submodules as $S(0)$-locally second submodules. We introduce the notions of $S(0)$-weakly second submodules over commutative rings. We complete this work by giving example $S(0)$-weakly second submodules which is not a weakly second submodule, characterizing the $S(0)$-weakly second submodules and proving some properties of $S(0)$-weakly second submodules related to multiplication and comultiplication modules.

2. $S(0)$-Weakly Second Submodules

Throughout this paper, rings mean associative commutative rings with unit and modules mean left R-modules. We consider first the following notions. Let R be a ring, N a submodule of a left R-module M. Denote

$$S(N) = \{r \in R \,|\, rm \in N \text{ for some } m \in M\backslash N\}.$$

If $N = 0$, then we have

$$S(0) = \{r \in R \,|\, rm = 0 \text{ for some } m \in M\backslash 0\}.$$

Lemma 2.1. *Let N be a nonzero submodule of M, P a maximal ideal of R with $S(0) \subseteq P$, and N_P is a submodule of M_P. Then $N_P \neq 0$.*

Proof. Suppose that $N_P = 0$. Let $0 \neq n \in N$, then it is clear that $0 = \frac{n}{p} \in N_P$. Then there exists $q \notin P$ such that $qn = 0p = 0$. Since $S(0) \subseteq P$, we obtain $q \notin S(0)$. Consequently, we get $n = 0$, a contradiction. Therefore, $N_P \neq 0$. □

Lemma 2.2. *Let M be an R-module, P a maximal ideal of R satisfying $S(0) \subseteq P$, and N a weakly second submodule of M. Then N_P is a weakly second submodule of M_P.*

Proof. Since $N \neq 0$, we get $N_P \neq 0$. Let $\frac{r}{p_1}, \frac{s}{p_2} \in R_P$ and a submodule K_P in M_P such that $\frac{r}{p_1}\frac{s}{p_2}N_P \subseteq K_P$ or $(rsN)_P \subseteq K_P$. Consequently, $rsN \subseteq K$. Since N is a second submodule of M, $rN \subseteq K$ or $sN \subseteq K$. Thus, $(rN)_P \subseteq K_P$ or $(sN)_P \subseteq K_P$. Equivalently $\frac{r}{p_1}N_P \subseteq K_P$ or $\frac{s}{p_2}N_P \subseteq K_P$. We prove that N_P is a second submodule of M_P. □

There exists a submodule N which is not a weakly second submodule in M but the N_P is a weakly second submodule in M_P as given in the example below.

Example 2.1. The $\mathbb{Z}$-submodule $2\mathbb{Z}$ is not a weakly second submodule, since $2\mathbb{Z}/8\mathbb{Z} \simeq \mathbb{Z}_4$ and $\mathrm{Ann}_{\mathbb{Z}}(2\mathbb{Z}/8\mathbb{Z}) = 4\mathbb{Z}$ is not a prime ideal in $\mathbb{Z}$. We consider the prime ideal $P = \{0\}$ of $\mathbb{Z}$, the multiplicative closed set in $\mathbb{Z}$ is $\mathbb{Z}\backslash\{0\}$ and $\mathbb{Z}_P = \mathbb{Q}$ is the fraction of module of $\mathbb{Z}$. Moreover, $(2\mathbb{Z})_P = (8\mathbb{Z})_P = \mathbb{Q}$ and $\mathrm{Ann}_{\mathbb{Q}}((2\mathbb{Z})_P/(8\mathbb{Z})_P) = \mathbb{Q}$, a prime ideal in $\mathbb{Q}$ itself. Thus, $(2\mathbb{Z})_P$ is a weakly second submodule of $(\mathbb{Z})_P = \mathbb{Q}$.

The definition of $S(0)$-weakly second submodules is given below.

Definition 2.1. Let M be an R-module and P a prime ideal of R satisfying $S(0) \subseteq P$. A nonzero submodules N of M is called an $S(0)$-weakly second submodule if N_P is a weakly second submodule.

Now we give a proposition about the sufficient and necessary condition for a nonzero submodule to become an $S(0)$-weakly second submodule.

Proposition 2.1. *Let M be an R-module, N a nonzero submodule of M, and P a maximal ideal R where $S(0) \subseteq P$. N is a $S(0)$-weakly second submodule of M if and only if for all $r, t \in R$, $rtN = tN$ or $rtN = rN$.*

Proof. (⇒) Since N is a $S(0)$-weakly second submodule of M, N_P is a weakly second submodule of M_P. Let $\frac{r}{p_1}, \frac{t}{p_2} \in R_P$ with $\frac{r}{p_1}\frac{t}{p_2}N_P \subseteq \frac{r}{p_1}\frac{t}{p_2}N_P$. We obtain $\frac{r}{p_1}N_P \subseteq \frac{r}{p_1}\frac{t}{p_2}N_P$ or $\frac{t}{p_2}N_P \subseteq \frac{r}{p_1}\frac{t}{p_2}N_P$. Equivalently $(rN)_P \subseteq (rtN)_P$ or $(tN)_P \subseteq (rtN)_P$. Since N_P is a weakly second submodule of M_P, we have $(rtN)_P \subseteq (rN)_P$ or $(rtN)_P \subseteq (tN)_P$. Thus, $(rtN)_P = (rN)_P$ or $(rtN)_P = (tN)_P$. Hence $rtN = rN$ or $rtN = tN$ for all $r, t \in R$.

(⇐) Let $\frac{r}{p_1}, \frac{t}{p_2} \in R_P$ and a submodule K_P in M_P such that $\frac{r}{p_1}\frac{t}{p_2}N_P \subseteq K_P$. It implies $(rtN)_P \subseteq K_P$ or $rtN \subseteq K$, for all $r, t \in R$. Based on the hypothesis then we get $tN \subseteq K$ or $rN \subseteq K$. Then, $(tN)_P \subseteq K_P$ or $(rN)_P \subseteq K_P$. Hence, we obtain $\frac{t}{p_2}N_P \subseteq K_P$ or $\frac{r}{p_1}N_P \subseteq K_P$. Thus, N_P is a weakly second submodule of M_P. Then N is a $S(0)$-weakly second submodule of M. □

As direct consequences, we obtain the following.

Corollary 2.1. *Let M be an R-module, N a nonzero submodule M, P a prime ideal of R where $S(0) \subseteq P$. N is an $S(0)$-weakly second submodule of M if and only if for any ideal I and J of R, $IJN = IN$ or $IJN = JN$.*

Corollary 2.2. *Let M be an R-module, N a nonzero submodule M, P a prime ideal of R where $S(0) \subseteq P$. N is a $S(0)$-weakly second submodule of M if and only if for any ideal I, J of R and a submodule K of M with $IJN \subseteq K$, $IN \subseteq K$ or $JN \subseteq K$.*

Now, we give the characterization of $S(0)$-weakly second modules.

Proposition 2.2. *Let M be an R-module, N a nonzero submodule M, P a prime ideal of R where $S(0) \subseteq P$. Suppose $\mathrm{Ann}_{R_P}(N_P)$ is a prime ideal of R_P. The following assertions are equivalent:*

(a) *N is a $S(0)$-weakly second submodule of M;*
(b) *For any ideal I in R_P, $IN_P = N_P$ or IN_P is a product of N_P and a prime ideal of R_P;*
(c) *$IN_P = N_P$ for any ideal I in R_P which is not prime;*
(d) *$IN_P = N_P$ for any ideal I in R_P which containing a proper prime ideal.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(1) $\Rightarrow$ (2) Since N is a $S(0)$-weakly second submodule of M, then N_P is a weakly second submodule of M_P. Moreover, we have $IN_P = N_P$ or INP is a product of N_P and a prime ideal of R_P.

(4) $\Rightarrow$ (1) Take any proper submodule K_P of N_P and we set $C = \mathrm{Ann}_{R_P}(N_P/K_P)$. We will show that C is a prime ideal of R_P. Then $\mathrm{Ann}_{R_P}(N_P) \subseteq C$ and $CN_P \subseteq K_P \neq N_P$. Suppose that $\mathrm{Ann}_{R_P}(N_P) \subset C$. Based on the hypothesis, C contains a proper prime ideal. We have $CN_P = N_P$, a contradiction since $CN_P \subseteq K_P \neq N_P$. Thus, $\mathrm{Ann}_{R_P}(N_P) = C$. Hence, C is a prime ideal of R_P and N is a $S(0)$-weakly second submodule of M. $\square$

The other characteristics of $S(0)$-weakly second submodules are given below.

Proposition 2.3. *Let M be an R-module, N a nonzero submodule M, P a prime ideal of R where $S(0) \subseteq P$. N is a $S(0)$-weakly second submodule of M if and only if for any submodule $K \subset N$ of M, $\mathrm{Ann}_R(N/K)$ is a prime ideal of R.*

Proof. ($\Rightarrow$) Since N is a $S(0)$-weakly second submodule of M, N_P is a weakly second submodule of M_P and $\mathrm{Ann}_{R_P}(N_P/K_P)$ is a prime ideal of R_P. Let $x, y \in \mathrm{Ann}_R(N/K)$ with $xy \in \mathrm{Ann}_R(N/K)$. It implies $xyN \subseteq K$ or $(xyN)_P \subseteq K_P$. Thus, $\frac{x}{p_1}\frac{y}{p_2}N_P \subseteq K_P$. Since $\mathrm{Ann}_{R_P}(N_P/K_P)$ is a prime ideal, we obtain $\frac{x}{p_1}N_P \subseteq K_P$ or $\frac{y}{p_2}N_P \subseteq K_P$. Consequently, we get $(xN)_P \subseteq K_P$ or $(yN)_P \subseteq K_P$. Thus, $xN \subseteq K$ or $yN \subseteq K$. Hence $\mathrm{Ann}_R(N/K)$ is a prime ideal of R.

($\Leftarrow$) Let $\frac{x}{p_1}, \frac{y}{p_2} \in \mathrm{Ann}_{R_P}(N_P/K_P)$ with $\frac{x}{p_1}\frac{y}{p_2} \in \mathrm{Ann}_{R_P}(N_P/K_P)$. It means $\frac{x}{p_1}\frac{y}{p_2}N_P \subseteq K_P$ or $(xyN)_P \subseteq K_P$. We get $xyN \subseteq K$. Since $\mathrm{Ann}_R(N/K)$ is a prime ideal, $xN \subseteq K$ or $yN \subseteq K$. Therefore, $(xN)_P \subseteq K_P$ or $(yN)_P \subseteq K_P$.

So $\frac{x}{p_1}N_P \subseteq K_P$ or $\frac{y}{p_2}N_P \subseteq K_P$. We obtain $\frac{x}{p_1} \in \text{Ann}_{R_P}(N_P/K_P)$ or $\frac{y}{p_2} \in \text{Ann}_{R_P}(N_P/K_P)$. Hence $\text{Ann}_{R_P}(N_P/K_P)$ is a prime ideal of R_P. Thus, N_P is a weakly second submodule of M_P or N is a $S(0)$-weakly second submodule of M. □

Proposition 2.4. *Let M be an R-module, N a nonzero submodule M, P a prime ideal of R where $S(0) \subseteq P$. If N is an $S(0)$-weakly second submodule of M then $\text{Ann}_R(N)$ is a prime ideal of R.*

Proof. Let $r, t \in R$ satisfying $rt \in \text{Ann}_R(N)$, then $rtN = 0$. Therefore, we obtain $(rtN)_P = 0$. Since N is an $S(0)$-weakly second submodule of M, then we obtain $(rN)_P = 0$ or $(tN)_P = 0$. Consequently, $rN = 0$ or $tN = 0$, so we get $r \in \text{Ann}_R(N)$ or $t \in \text{Ann}_R(N)$. Hence $\text{Ann}_R(N)$ is a prime ideal of R. □

Proposition 2.5. *Let M be an R-module and P a maximal ideal of R where $S(0) \subseteq P$. If M is an $S(0)$-weakly second module then M/L is also an $S(0)$-weakly second module, for each proper submodule L of M.*

Proof. Let L be a proper submodule of M. We will show that M_P/L_P is a weakly second module. Let $\frac{r}{p_1}, \frac{t}{p_2} \in R_P$ and submodule K_P of M_P with $L_P \subseteq K_P$ and $\frac{r}{p_1}\frac{t}{p_2}M_P/L_P \subseteq K_P/L_P$. Then $(rtM)_P/L_P \subseteq K_P/L_P$. So, $(rtM)_P \subseteq K_P$. Since M_P is a weakly second module, $(rM)_P \subseteq K_P$ or $(tM)_P \subseteq K_P$. Therefore, $(rM + L)_P/L_P \subseteq K_P/L_P$ or $(tM + L)_P/l_P \subseteq K_P/L_P$. Therefore, $\frac{r}{p_1}M_P/L_P \subseteq K_P/L_P$ or $\frac{t}{p_2}M_P/L_P \subseteq K_P/L_P$. Hence M/L is an $S(0)$-weakly second module. □

Proposition 2.6. *Let M be a nonzero R-module and P a prime ideal of R where $S(0) \subseteq P$. If M_P is a comultiplication R_P-module and N is an $S(0)$-weakly second submodule of M, then N_P is a second submodule of M_P.*

Proof. Since N is an $S(0)$-weakly second submodule of M, then N_P is a weakly second submodule of M_P. Let $\frac{r}{p_1} \in R_P$ and L_P is a completely irreducible submodule of M_P satisfying $\frac{r}{p_1}N_P \subseteq L_P$. Since M_P is a comultiplication module, then there exists an ideal I_P in R_P such that $L_P = (0_P :_{M_P} I_P)$. Thus, $\frac{r}{p_1}N_P \subseteq (0_P :_{M_P} I_P)$ or equivalent to say $\frac{r}{p_1}I_PN_P = 0_P$. Since N_P is a weakly second submodule of M_P, $\text{Ann}_{R_P}(N_P)$ is a prime ideal of R_P. Thus, $\frac{r}{p_1}N_P = 0_P$ or $I_PN_P = 0_P$. Equivalently, $\frac{r}{p_1}N_P = 0_P$ or $N_P \subseteq (0_P :_{M_P} I_P) = L$. Hence, N_P is a second submodule of M_P. Thus, N is an $S(0)$-locally second submodule of M. □

Proposition 2.7. *Let M be a nonzero R-module and P a prime ideal of R where $S(0) \subseteq P$. If M_P is a multiplication R_P-module and M is weakly second R_P-module, then M is an $S(0)$-weakly prime R-module.*

Proof. Let $\frac{r}{p_1}, \frac{s}{p_2} \in R_P$. Since M is an $S(0)$-weakly second R-module, then M_P is a weakly second R_P module. So, we get $\frac{r}{p_1}\frac{s}{p_2}M = \frac{r}{p_1}M$. Since M_P is a multiplication module, this implies that

$$\begin{aligned}0 &= \frac{r}{p_1}\frac{s}{p_2}\left(\left(0 :_M \frac{r}{p_1}\frac{s}{p_2}\right) :_{R_P} M_P\right) M_P \\ &= \left(\left(0 :_M \frac{r}{p_1}\frac{s}{p_2}\right) :_{R_P} M_P\right) \frac{r}{p_1} M_P \\ &= \left(0 :_{M_P} \frac{r}{p_1}\frac{s}{p_2}\right) \frac{r}{p_1}.\end{aligned}$$

It follows that $(0 : M_P \frac{r}{p_1}\frac{s}{p_2}) = (0 :_{M_P} \frac{r}{p_1})$. Thus, M_P is a weakly prime R_P-module, i.e. M is an $S(0)$-weakly prime R-module. □

Proposition 2.8. *Let M be a nonzero R-module and P a prime ideal of R satisfying $S(0) \subseteq P$. If M_P is a comultiplication R_P-module and M is an $S(0)$-weakly prime R-module, then M is an $S(0)$-weakly second R-module.*

Proof. Let $\frac{r}{p_1}, \frac{s}{p_2} \in R_P$. Since M is an $S(0)$-weakly prime R-module, then M_P is a weakly prime R_P module. So, we get $(0 : M_P \frac{r}{p_1}\frac{s}{p_2}) = (0 :_{M_P} \frac{r}{p_1})$. Since M_P is a comultiplication module, this implies that

$$\begin{aligned}M_P &= \left(\left(0 :_{M_P} \frac{r}{p_1}\frac{s}{p_2}\right) :_{M_P} \mathrm{Ann}_{R_P}\left(\frac{r}{p_1}\frac{s}{p_2} M_P\right)\right) \\ &= \left(\left(0 :_{M_P} \frac{r}{p_1}\right) :_{M_P} \mathrm{Ann}_{R_P}\left(\frac{r}{p_1}\frac{s}{p_2} M_P\right)\right) \\ &= \left(\frac{r}{p_1}\frac{s}{p_2} M_P :_{M_P} \frac{r}{p_1}\right).\end{aligned}$$

It follows that $\frac{r}{p_1}\frac{s}{p_2} M = \frac{r}{p_1} M$. Thus, M_P is a weakly second R_P-module. □

Acknowledgment

The authors thank to the referees for valuable comments and suggestions. This work is supported by Gant Penelitian Berbasis Kompeteusi, Directurate General of Higher Education, Department of Research, Technology and Higher Education Indonesia, 2017.

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