Solving Stochastic Linear Quadratic Games in Discrete Time with Two Players Using Exact Line-Double Newton Method

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***Abstract*—in this paper we consider a two-player stochastic linear quadratic differential games with an infinite horizon in discrete time. We assumed that there is no cooperation between the two players. For the given system, the major problem is solving a pair of stochastic discrete algebraic Riccati equation (SDAR) arising on the given system and its quadratic regulator to find its optimal control form. Thus, we construct a numerical method for solving SDAR using a modified Newton method. This method is modified from its original Newton method and Exact line search method to concise the Newton iteration. We provide a numerical simulation to show the performance of the method.**

***Index Terms*—modified Newton, algebraic Riccati, discrete stochastic game**

I. Introduction

The study of decision-making strategies has become an interesting topic among researchers using the dynamic game theory. This is obtained by examining the linear quadratic regulator with its differential system. The results of this work can be found in [1]– [7].

On the dynamical game theory, two players are involved to make some decisions by minimizing their performance index and are given control as desired to the linear quadratic game. To find the solution of linear quadratic game means to find all the possible equilibrium solutions and the stabilizing solution of the algebraic Riccati equation. Further, we consider a generalized algebraic Riccati equation of linear quadratic stochastic games involving two players. It is assumed that there is no cooperation between the leader and the follower. This will obtain a couple of algebraic Riccati equations such as describe in [1], [2], [4] and [7]. Since the algebraic Riccati equations are difficult to be solved, we adopt the work in [8] to solve the algebraic Riccati equation in stochastic control. Then we can extend this method to the couple of algebraic Riccati equations in stochastic games.

In [4] the linear quadratic differential games were analyzed for an infinite planning horizon. It is also assumed that each player formulates their strategy at the beginning of the system, resulting in the players inability to change the strategy during the system run. Further in [4] a set of coupled algebraic Riccati equation related to the system and its performance index. Then it can find all the equilibriums of the system under some consideration. In the other works, [6] observe the properties to solve discrete algebraic Riccati equation in an open loop Nash and Stackelberg games with non-cooperative condition.

The difficulty encountered in solving a coupled of algebraic Riccati equations is that the player who acts as a leader involves the control results from the follower. As a result, the Hamiltonian matrix of each player involves the control results of the other players. Based on these constraints, a method can be built to solve the coupled algebraic Riccati using a numerical method. There are several references that describe the numerical method for solving Riccati algebraic equations for both discrete and continuous cases using the Newton method and its modifications.

Newton's method and its modifications applied to solve the algebraic Riccati equations can be found in [8] and [9] for a solution to the algebraic Riccati equations in the stochastic case. While [10] applied the modified Newton method to solve the algebraic Riccati equations and [11] constructing some numerical method in solving Algebraic Riccati equation for both continuous and discrete case. In other study, the Newton method in [2] was used to solve the Riccati problem that appeared in the Nash and Stackelberg open-loop discrete time game. Furthermore, the numerical method for solving the Nash equilibrium of the Riccati pair of algebraic equations in multi-model systems can be found in [5]. The exact-line search method is also used to solve non-symmetric algebraic Riccati equations in open-loop linear differential games (see [7] and [12]). In [13] generating an algorithm for solving LQ difference games with multiplication noices and Marcovian jump for discrete time.

In several previous studies, research has been carried out in finding the optimal solution of dynamic two-player games using the Newton method in [10]. Meanwhile, no research has been conducted to find optimal solutions for games in stochastic game systems. Therefore, in this study, we are

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interested in examining the optimal Nash solution that arises from stochastic games. Further, the modified Newton method is used to solve the paired of algebraic Riccati equations that appear in the stochastic differential games were obtained. In this case the players non-cooperate with each other.

In this paper, we give a two-player stochastic game formulation in the second section. In the next section a numerical method is given to solve the algebraic Riccati equations that appear from its linear quadratic regulator. At the end of the discussion, a numerical simulation is given to solve the algebraic Riccati equation using the modified Newton method.

II. Problem Formulation

A stochastic game is a combination of the dynamic game theory and stochastic systems. Stochastic games are also equipped with objective functions that will be minimized by each player involved in it. The objective function given to the system is a linear quadratic regulator involving state vectors and control vectors. In this case, we work on the discrete time system called as the discrete stochastic linear quadratic differential games with infinite horizon involving two players where the system is defined by the following form

$$x_{k+1} = Ax_k + \sum_{i=1}^{2} B_i u_{i,k} + \omega_k \left[Cx_k + \sum_{i=1}^{2} D_i u_{i,k} \right]; \quad k = 0,1,2,... \tag{1}$$

with initial state x_0, where the matrices $A, C \in \mathbb{R}^{n\times n}$, $B_1, D_1 \in \mathbb{R}^{n\times m_1}$, and $B_2, D_2 \in \mathbb{R}^{n\times m_2}$, x_k is an n-dimensional vector, while $u_{i,k}; i = 1,2$ are n-dimensional control vector which is corresponding to the i-th player. The disturbance ω_k defined as random variable with mean $\omega_k = 0$ and variance $\omega_k \omega_k^T = \alpha_{i,k}$.

For the given system, the form of the objective function to be minimized by the players is given by the following expression

$$J_i(u_1, u_2) = \sum_{k=0}^{\infty} \frac{\xi}{2} \left[x_k^T Q_i x_k + u_{1,k}^T R_{1i} u_{1,k} + u_{2,k}^T R_{2i} u_{2,k} \right] \tag{2}$$

where $Q_i, R_{1,k}, R_{2,k}$ are symmetric matrices with $n \times n, m_1 \times m_1, m_2 \times m_2$ dimensions respectively. The estimator operator given by ξ in (2) serve as stochastic process on the objective function.

Thus, a coupled of algebraic Riccati equation produced by (1) and (2) can be written as follows

$$\begin{aligned} &-X_i + \hat{A}^T X_i \hat{A} + \hat{C}^T X_i \hat{C} + Q_i^T Q_i + L_i^T R_{ii} L_i \\ &-\left[B_j^T X_i \hat{A} + D_j^T X_i \hat{C} \right]^T \hat{R}^{-1} \left[B_j^T X_i \hat{A} + D_j^T X_i \hat{C} \right] = 0 \end{aligned} \tag{3}$$

where

$$\hat{A} = A - B_i K_i, \quad \hat{R} = R_{ji} + B_j^T X_i B_j + D_j^T X_i D_j,$$
$$\hat{C} = C - D_i K_i, \quad j \neq i, i = 1,2.$$

The equilibrium action for the players is given by

$$u_k^* = -\hat{R}^{-1} \left[B_j^T X_{i,k} \hat{A} + D_j^T X_{i,k} \hat{C} \right]. \tag{4}$$

In this game, we assumed that the players interact with each other. The set of actions $u^* = (u_1^*, u_2^*)$ satisfying (1) can be called as Nash equilibrium if it satisfies $J_i(x_0, u_1^*, u_2^*) \leq J_1(x_0, u_1, u_2^*)$ and $J_i(x_0, u_1^*, u_2^*) \leq J_1(x_0, u_1{}^*, u_2)$ for each state feedback matrix $u_i; i = 1,2$.

Since solving (3) is quite difficult analytically, we can solve (3) using numerical approximation. The method will be discussed on the next section.

III. Exact-Line Double Newton Method

A numerical method is used in order to solve a couple of algebraic Riccati equations arising on SDAR (1). The most popular method for solving algebraic Riccati equations is the Newton method which can be found in [11]. Then, the method is developed in [3], [7] and [12] for solving a couple of algebraic Riccati equations that arise on linear quadratic games continuous time. Moreover, a numerical method applies in solving a discrete algebraic Riccati equation arise on open loop Nash and Stackelberg games conducted by [2].

In this paper, we adopt a modified Newton method using Exact Line Search in [11] and Doubled Newton method in [11]. We then develop the modified Newton method on [10] in solving discrete algebraic Riccati equation rise on the stochastic Linear Quadratic Differential games (1). This is done because of the significance of the Newton method and its modification in solving discrete Riccati algebraic equations as described on [7]– [13].

As it is mentioned in [7] and [10] the Newton method needs a stabilizing strategy u_i for the players in the first step, such that the pair of matrices (A, B) and (C, D) in (1) are stable. Given a Stein equation

$$\begin{aligned} &X_{i,0} - A_{i,0}^T X_{i,0} A_{i,0} - C_{i,0}^T X_{i,0} C_{i,0} \\ &= Q^T Q + L_0^T R_{ii} L_0 - C_i{}^T L_0 - L_0^T C_i. \end{aligned} \tag{5}$$

with $A_{i,0} = A_i - B_i L_0$ and $C_{i,0} = C_i - D_i L_0$. The initial guess matrix $X_{i,0}$ can be found by solving (5).

Further, taking a coupled of Newton step size $H_{i,k} = X_{i,k-1} - X_{i,k}$ into the Newton step such that we have a Newton iteration

$$\Re(X_{i,k}) + H_{i,k} - A_k{}^T H_{i,k} A_k - C_k{}^T H_{i,k} C_k = 0, \tag{6}$$

where

$$\begin{aligned} L_k &= [R_{ji} + \Gamma_k]^{-1} \left[B_j^T X_{i,k} A_k + D_j^T X_{i,k} C_k \right] \\ \Gamma_k &= B_j^T X_{i,k} B_j + D_j^T X_{i,k} D_j \\ A_k &= A_i - B_i L_k \\ C_k &= C_i - D_i L_k, \end{aligned}$$

and $\Re(X_{i,k})$ is the left hand side of (3).

While the Double Newton step is a modification of Newton method where the step size of this method can be replaced with $2H_{i,k} = X_{i,k-1} - X_{i,k}$.

We now define the Exact Line step by taking the step size $H_{i,k} = X_{i,k-1} - t_{i,k}X_{i,k}$ and substitute it into (3). Thus we have

$$\Re\left(X_{i,k} - t_{i,k}H_{i,k}\right) = \left(1 - t_{i,k}\right)\Re\left(X_{i,k}\right) - t_{i,k}^2\Upsilon_k + O\left(t^3\right), \quad (7)$$

under some consideration with

$$\Upsilon_k = A_k^T H_{i,k}G_{i,k}H_{i,k}A_k + C_k^T H_{i,k}G_{i,k}H_{i,k}C_k$$
$$G_{i,k} = B_k^T\left[R_{ji} + \Gamma_k\right]^{-1}$$

Further, the minimum value of t can be found by minimizing a *Frobenius* norm of (7), that is

$$\min\ f = \left\|\Re\left(X_{i,k} - t_{i,k}H_{i,k}\right)\right\|_F^2. \quad (8)$$

Thus, the t value obtained here is used to iterate on Exact Line step.

Thus, we can construct a modified method for solving SDAR using Exact Line and Double Newton step. The method can be written as follows.

Algorithm

1) Choose initial L_0 such that $A - BL_0$ and $C - DL_0$ stable, (see [11] and [12] to find a stabilizing matrix L_0);
2) Find $X_{i,0}$ satisfying (5);
3) For $k \geq 0$
   a) Solve (6) with $H_{i,k} = X_{i,k-1} - X_{i,k}$;
   b) Find $t_{i,k}$ satisfying (8);
   c) Compute $X_{i,k+1} = X_{i,k} - t_{i,k}H_{i,k}$;
   d) If $t_{i,k}$ satisfy $\left\|\Re\left(X_{i,k} - t_{i,k}H_{i,k}\right)\right\| < \varepsilon$ then stop the iteration;
   e) Otherwise compute $X_{i,k+1} = X_{i,k} - 2H_{i,k}$;
   f) If $\left\|\Re\left(X_{i,k+1}\right)\right\| < \sqrt{1 - 2t_{i,k}}\left\|\Re\left(X_{i,k}\right)\right\|$ then go to 3.h;
   g) Otherwise $X_{i,k+1} = X_{i,k} - H_{i,k}$;
   h) If $\left\|\Re\left(X_{i,k+1}\right)\right\| < \varepsilon$ then stop the iteration;
   i) Else, go to 3.b.

IV. Numerical Example

In this section, a numerical example is given to provide a simulation in solving SDAR using an Exact Line-Double Newton method. According to the system (1), we consider the example with a system involving to two players and a free random variable. The system can be defined as

$$x_{k+1} = Ax_k + B_1u_{1,k} + B_2u_{2,k} + \omega_k\left[Cx_k + D_1u_{1,k} + D_2u_{2,k}\right]$$

with

$$A = \begin{bmatrix}-1 & 0\\ 0.5 & 0.002\end{bmatrix}, B_1 = \begin{bmatrix}1 & 0\\ -0.1 & 1\end{bmatrix},$$
$$B_2 = \begin{bmatrix}0.01 & -1\\ 0.5 & 0.01\end{bmatrix}, C = \begin{bmatrix}0.05 & 0.01\\ 0.01 & -0.001\end{bmatrix},$$
$$D_1 = \begin{bmatrix}0.01 & 0.001\\ 0.01 & -0.001\end{bmatrix}, D_2 = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}.$$

The quadratic regulator function to be minimized has the form

$$J_1(u) = \xi\sum_{k=0}^{\infty} x_k^2 + u_{1,k}^T\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}u_{1,k} + u_{2,k}^T\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}u_{2,k}$$

and

$$J_2(u) = \xi\sum_{k=0}^{\infty} x_k^2 + u_{1,k}^T\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}u_{1,k} + u_{2,k}^T\begin{bmatrix}1 & 0.5\\ -1 & 1\end{bmatrix}u_{2,k}.$$

First of all, we take a stopping criterion $\varepsilon = 10^{-10}$ for stopping the iteration. For the above problem, we have a stabilizing matrix

$$L_0 = \begin{bmatrix}-0.0487 & -0.0001\\ 0.0305 & -0.0002\\ -0.2550 & -0.0015\\ -0.6687 & -0.0041\end{bmatrix}.$$

Thus we can take an initial guess

$$X_0 = \begin{bmatrix}-0.0675 & -0.0900\\ -0.0149 & 0.0755\\ -0.7291 & -2.0410\\ 0.1096 & 0.3519\end{bmatrix}.$$

Below, we provide a summarizing iteration of Exact Line-Double Newton method in solving the above problem.

TABLE I
TABLE OF EXACT LINE-DOUBLE NEWTON ITERATION

k	i	X_k	t_k	$\|\Re(X_k)\|$
0	1	$\begin{bmatrix}0.9569 & -0.0086\\ 0.0230 & 1.0045\end{bmatrix}$	–	0.7008
	2	$\begin{bmatrix}-0.0675 & -0.09\\ -0.0149 & 0.0755\end{bmatrix}$		
1	1	$\begin{bmatrix}4.9629 & -0.0205\\ -0.0011 & 0.9951\end{bmatrix}$	1.1101	0.4051
	2	$\begin{bmatrix}2.3538 & 0.0078\\ 0.0091 & 0.9998\end{bmatrix}$		
2	1	$\begin{bmatrix}3.3829 & 0.0248\\ 0.0185 & 1.0001\end{bmatrix}$	1.0019	0.0066
	2	$\begin{bmatrix}2.3553 & -0.0201\\ -0.0204 & 0.9953\end{bmatrix}$		
⋮	⋮	⋮	⋮	⋮
12	1	$\begin{bmatrix}3.3403 & 0.027\\ 0.019 & 1.0002\end{bmatrix}$	1	5.9202×10^{-12}
	2	$\begin{bmatrix}2.2484 & 0.0048\\ 0.0098 & 1.0002\end{bmatrix}$		

[a]Using software matlab.

The above table shows that it takes 12 iterations to solve the problem using Exact Line-Double Newton methods. The optimal solution of the game is

$$X = \begin{bmatrix}3.3403 & 0.027\\ 0.019 & 1.0002\\ 2.2484 & 0.0048\\ 0.0098 & 1.0002\end{bmatrix}.$$

The step size used in the last iteration is

$$H_k = 10^{-11}\begin{bmatrix} -0.5270 & -0.1789 \\ -0.0191 & -0.0013 \\ -0.3580 & -0.0160 \\ -0.0054 & -0.0001 \end{bmatrix}$$

with $\|\Re(X_k)\| = 5.9209 \times 10^{-12}$ which less than the specified error.

V. Conclution

Exact Line-Double Newton method is a modification method constructed from its original Newton method by combining the Exact Line search method and the Double Newton method. This method can be applied to solve Stochastics discrete algebraic Riccati equations rising on two-player games with linear quadratic Regulator. It needs an initial guess matrix to start the iteration. It also guarantees to produce a stabilizing solution. Since the Newton method and the Exact Line search method converge, intuitively Exact Line-Double Newton method also converge. The convergence of this method will be conducted in the next discussion.

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