On Lambda Class Modules

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ON A CLASS OF λ-MODULES

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In [*Int. Electron. J. Algebra*, **15**, 173 (2014)], Smith introduced maps between the lattice of ideals of a commutative ring and the lattice of submodules of an R-module M, i.e., μ and λ mappings. The definitions of the maps were motivated by the definition of multiplication modules. Moreover, some sufficient conditions for the maps to be lattice homomorphisms were studied. We define a class of λ-modules and indicate the properties of this class. We also present sufficient conditions for the module and the ring under which the class λ is a hereditary pretorsion class.

1. Introduction

A ring R is understood as any commutative ring with unit and a module M is a left R-module, unless otherwise stated. An R-module M is called a multiplication module if, for any submodule N in M, there is an ideal I in R such that $N = IM$. For further explanation of multiplication modules over commutative rings, we refer the reader to [4, 8, 13]. Moreover, M is a multiplication module if and only if, for any submodule N of M, we have $N = \mathrm{Ann}_R(M/N)M$ (see [8]).

An R-module M is called a prime module if, for any nonzero submodule K in M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$. A proper submodule N in M is called a prime submodule of M if M/N is a prime module (see [14]).

Let K and N be submodules of M. The residue of K in N is denoted by

$$[N :_R K] = \{r \in R \mid rK \subseteq N\}.$$

For a special case where $N = 0$, we obtain the annihilator of K as follows: $[0 :_R K] = \mathrm{Ann}_R(K)$.

Let $\mathcal{L}(M)$ be the lattice of submodules of the R-module M, where for any submodules N and K in M, the "join" and "meet" are defined as

$$N \vee K = N + K \quad \text{and} \quad N \wedge K = N \cap K,$$

and $N \leq K$ means that $N \subseteq K$. In particular, for $M = R$, we have the lattice of ideals in R, which is denoted by $\mathcal{L}(R)$. The definitions of μ and λ mappings introduced by Smith in [12] are as follows:

$$\mu\colon \mathcal{L}(M) \to \mathcal{L}(R), \quad N \mapsto \mathrm{Ann}_R(M/N), \tag{1.1}$$

$$\lambda\colon \mathcal{L}(R) \to \mathcal{L}(M), \quad I \mapsto IM. \tag{1.2}$$

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Mappings (1.1) and (1.2) are motivated by the relationship between submodules and ideals in a multiplication module. Thus, we define a class of modules as follows:

$$\lambda = \{M \mid (B\cap C)M = BM \cap CM \ \ \forall B, C \ \text{(finitely generated ideals of } R)\}.$$

By virtue of Lemma 2.1 of [12], $M \in \lambda$ if and only if M is a λ-module. Note that λ is not necessary a hereditary class.

If R is a ring, then an R-module M is called a chain module if, for any submodules N and L in M, either $N \subseteq L$ or $L \subseteq N$. A ring R is called a chain ring if the R-module R is a chain module. In Proposition 2.4 of [12], Smith proved the following sufficient condition for a ring under which its modules are in λ:

Proposition 1.1. *If a ring R is a chain ring, then every R-module is in λ.*

Moreover, the class λ is closed under direct summands and direct sums (see Lemma 2.5 of [12]). Theorem 2.3 in [12] gives a necessary and sufficient condition for a module to be in λ. We now recall this result.

Proposition 1.2. *The following assertions are equivalent:*

(a) *R is Prüfer;*

(b) *every R-module is in λ;*

(c) *the class λ is closed under the homomorphic image.*

The sufficient conditions in Proposition 2.4 and Theorem 2.3 from [12] motivated us to study more general situations from category R-modules R-Mod to subcategory $\sigma[M]$, which consists of M-subgenerated modules. In the present paper, we show that, under certain additional conditions, if the subgenerator M is a Dedekind module or a chain module, then the class λ is equal to the class $\sigma[M]$.

In the next section, we discuss Dedekind modules and the relationship with the class λ. In Section 3, it is shown that Theorem 2.3 in [12] can be generalized.

2. Dedekind Modules and λ-Modules

For the extensive study of Dedekind modules, we refer the reader to Alkan, et al. [3] and Saraç, et al. [11]. For any commutative ring R with identity and a set S consisting of nonzero divisor elements of R, the fraction ring R_S is formed in a natural way. By analyzing the notion of fractional ideals introduced by Larsen and McCarthy [9], we conclude that a fractional ideal I of R is invertible if there exists a fractional ideal I^{-1} of R such that $I^{-1}I = R$. In the case where I^{-1} exists, we have $I^{-1} = [R :_{R_S} I]$. A domain R is called a Prüfer domain provided that each finitely generated ideal of R is invertible. Furthermore, an integral domain R is a Dedekind domain iff every nonzero ideal of R is invertible.

We now generalize the notion of invertibility of fractional ideals to the case of submodules. The notion of invertible submodules have been discussed in numerous papers (see, e.g., [3, 11]).

For any R-module M, consider

$$T = \{t \in S \mid \text{for some } m \in M,\ tm = 0 \text{ implies that } m = 0\}.$$

We can see that T is a multiplicatively closed subset of S. For any submodule N of M, we define

$$N' = [M :_{R_T} N].$$

Following the concept of invertible ideal, we say that a submodule N of M is invertible if $N'N = M$. Then M is called a Dedekind module if every nonzero submodule of M is invertible and M is called a Prüfer module provided that every finitely generated nonzero submodule is invertible. As examples of Dedekind modules, we can mention the $\mathbb{Z}$-module $\mathbb{Q}$ and $\mathbb{Z}_p$ for prime p.

An R-module M is called a multiplication module provided that, for each submodule N of M, there exist an ideal I of R such that $N = IM$, i.e., $I = [N :_R M]$. If P is a maximal ideal of R, then we define

$$T_P(M) = \{m \in M \mid (1-p)m = 0 \text{ for some } p \in P\}. \tag{2.1}$$

Further, M is P-cyclic if there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$. In 2007, El-Bast showed that M is a multiplication module if and only if, for every maximal ideal P of R, either $M = T_P(M)$ or M is P-cyclic.

We now establish the property of λ-module dealing with the invertibility property of submodules of multiplication modules. To do this, we recall an important property presented in [1], namely, for any finitely generated faithful multiplication R-module and any invertible submodule N of M, $[N :_R M]$ is an invertible ideal of R.

Proposition 2.1. *Let M be an R-module. Then the following assertions are true:*

1. *If I is a multiplication ideal of a ring R and M is a multiplication R-module, then $\lambda(I)$ is a multiplication R-module.*

2. *Every invertible submodule N of a faithful multiplication finitely generated module M is a λ-module.*

3. *If M is a faithful multiplication module over an integral domain R, then M is a λ-module and, for any ideal I of R, $I^{-1} = (\lambda(I))^{-1}$.*

Proof. 1. Let P be a maximal ideal of R. Consider the set T_P given by (2.1). If $T_P(M) = M$ or $T_P(I) = I$, then $T_P(IM) = IM$. Hence, $\lambda(I)$ is a multiplication module. Suppose that $T_P(I) \neq I$ and $T_P(M) \neq M$. Then I and M are P-cyclic. Therefore, there exist elements $p_1, p_2 \in P$, $a \in I$, $m \in M$ such that

$$(1-p_1)I \subseteq Ra \qquad \text{and} \qquad (1-p_2)M \subseteq Rm.$$

This implies that

$$(1-p)IM \subseteq R(am), \quad \text{where} \quad p = p_1 + p_2 - p_1p_2 \in P.$$

Thus, $\lambda(I) = IM$ is P-cyclic. This proves that $\lambda(I)$ is a multiplication R-module.

2. According to Proposition 2.1 in [1], for any invertible submodule N of M, $[N :_R M]$ is an invertible ideal of R. By using (2), we can easily show that $N = [N :_R M]M$ is a multiplication R-module. If $r \in \mathrm{Ann}_R(N)$, then $rN = 0$ and, hence, $rM = rN^{-1}N = 0$. Thus, $r = 0$. Therefore, N is a faithful multiplication R-module. By using Theorem 2.12 in [12], we conclude that N is a λ-module because every faithful multiplication module is a λ-module.

3. This assertion is obvious by Theorem 2.12 in [12] and Lemma 1 in [2].

We are now ready to deduce the relationship between the Dedekind module and the λ-module by using the following Corollary 3.8 in [3]. We recall that a module M is divisible if $M = rM$ for any $0 \neq r \in R$.

Lemma 2.1. *Let M be a Dedekind divisible R-module. Then R is a field.*

It is easy to understand that any vector space is a λ-module. Moreover, we have the following direct consequences of Lemma 2.1:

Proposition 2.2. *If M is a Dedekind divisible R-module, then:*

(1) M is λ-module;

(2) $N \in \lambda$ for any $N \in \sigma[M]$;

(3) the class of λ is closed under submodules and homomorphic images.

We now apply the following result from paper [1]:

Lemma 2.2. *If M is a faithful multiplication module, then M is a Dedekind (Prüfer) module if and only if R is a Dedekind (Prüfer) domain.*

Proposition 2.3. *Let M be a faithful multiplication module and a Prüfer module. Then $\sigma[M] \subseteq \lambda$.*

Proof. By the assumption and according to the result of Lemma 2.2, we conclude that R is a Prüfer domain. Proposition 1.2 shows that λ is equal to the category of R-modules. It is clear that $\sigma[M] \subseteq \Lambda$.

The converse assertion is given in the following corollary:

Corollary 2.1. *Let R be a semisimple ring and let M be a faithful multiplication module and a Prüfer module. If M is a subgenerator for any semisimple module, then $\sigma[M] = \lambda$.*

Proof. Applying Proposition 2.3, we get $\sigma[M] \subseteq \lambda$. Further, we take any $N \in \lambda$. Since R is a semisimple ring, N is also a semisimple module. Moreover, $N \in \sigma[M]$ and we also prove that $\lambda \subseteq \sigma[M]$.

We now recall a sufficient condition for a Dedekind module in Lemma 3.3 from [1].

Lemma 2.3. *Let R be an integral domain and let M be a faithful multiplication module. If any nonzero prime submodule P of M is invertible, then M is a Dedekind module.*

Proposition 2.4. *Let R be an integral domain and let M be a faithful multiplication module. If any nonzero prime submodule P of M is invertible, then $\sigma[M] \subseteq \lambda$.*

Proof. The proof is obvious if we apply Lemmas 2.3 and 2.2.

The next propositions establish some other properties of Dedekind modules.

Proposition 2.5. *Let M be a faithful multiplication Dedekind module over an integral domain R. If I is an ideal of R, then M is a λ-module over I and $\lambda(I)$ is a λ-module over R.*

Proof. Since I is an ideal of R, $\lambda(I) = IM$ is a submodule of M. Hence, IM is an invertible submodule. We have $I^{-1} = (\lambda(I))^{-1}$ by virtue of Proposition 2.1, i.e., I is an invertible ideal of R. By using a result from [7], we conclude that I is a λ-module over R. For any ideals B and C of R, we have

$$(B \cap C)\lambda(I) = (B \cap C)IM = (BI \cap CI)M$$

$$= BIM \cap CIM = B\lambda(I) \cap C\lambda(I).$$

This proves our assertion.

We now present a sufficient condition of λ-module.

Proposition 2.6. *Let M be a multiplication Dedekind R-module. Then every R-module is a λ-module.*

Proof. According to Theorem 3.12 in [3], a multiplication Dedekind R-module implies that the ring R is a Dedekind domain, i.e., a Prüfer domain. This means that every R-module is a λ-module.

If M is an R/I-module, then, under scalar multiplication $am = (a + I)m$, M becomes an R-module for every $a \in R$ and $m \in M$. Conversely, if M is an R-module, then M is an A/I-module with respect to $(a+I)m = am$ for every $a + I \in R/I$ and $m \in M$.

Proposition 2.7. *Let R be a ring, let M be an R-module, and let I be an ideal of R, where $I \subseteq [0 :_R M]$. Then M is a λ-module over R if and only if M is a λ-module over R/I.*

Proof. If M is a λ-module over R, then $(B \cap C)M = BM \cap CM$ for every finitely generated ideals B and C of R. Let B/I and C/I be any ideals of R. Then

$$(B/I \cap C/I)M = ((B \cap C)/I)M.$$

Since $(B \cap C)M = BM \cap CM$, we get

$$((B \cap C)/I)M = (B/I)M \cap (C/I)M.$$

This gives

$$(B/I \cap C/I)M = (B/I)M \cap (C/I)M.$$

Therefore, M is a λ-module over R/I.

Conversely, let M be a λ-module over R/I. Suppose that B and C are any ideals of R with $BM \neq \{0\}$ and $CM \neq \{0\}$. Clearly, $B + I/I$ and $C + I/I$ are ideals of R/I. Since M is a λ-module over R/I, we have

$$((B + I/I) \cap (C + I/I))M = (B + I/I)M \cap (C + I/I)M.$$

On the other hand,

$$((B + I/I) \cap (C + I/I))M = ((B + I \cap C + I)/I)M$$

and, consequently,

$$((B + I \cap C + I)/I)M = (B + I/I)M \cap (C + I/I)M.$$

Then

$$(B + I \cap C + I)M = (B + I)M \cap (C + I)M.$$

Since $I \subseteq (0 : M)$, we get

$$(B + I)M \cap (C + I)M = BM \cap CM.$$

Therefore, M is a λ-module over R.

3. Chain Modules and λ-Modules

In this section, we consider chain modules and the relationship with λ-modules.

Proposition 3.1. *Let M be a chain and faithful R-module. Then $\sigma[M] \subseteq \lambda$.*

Proof. For any $N \in \sigma[M]$, according to (15.1) in [15], we have

$$N = \oplus_\Lambda Rm_\lambda,$$

where $m_\lambda \in M^{(\mathbb{N})}$. Since M is chain, $N = Rm_0$ for some $m_0 \in M^{(\mathbb{N})}$. We now take any ideals B and C in the ring R. It is necessary to prove that

$$BN \cap CN \subseteq (B \cap C)N.$$

We take any $x \in BN \cap CN$. Then $x = bm_0$ for some $b \in B$ and $x = cm_0$ for some $c \in C$. Hence,

$$x = bm_0 = cm_0$$

and, moreover, $(b - c)m_0 = 0$. Since M is faithful, $b = c$ and we conclude that $x \in (B \cap C)N$.

For the converse of Proposition 3.1, we need an additional condition formulated in the following corollary:

Corollary 3.1. *Let R be a semisimple ring and let M be a chain and faithful R-module and a subgenerator for all semisimple R-modules. Then $\sigma[M] = \lambda$.*

Proof. We apply Proposition 3.1. It is known that a module over a semisimple ring is semisimple. We take any R-module N in λ. Then N is semisimple. By the assumption, $N \in \sigma[M]$.

According to the properties of $\sigma[M]$ from (15.1) in [15], we obtain the following corollary:

Corollary 3.2. *Let R be a semisimple ring and let M be a chain and faithful R-module and a subgenerator for all semisimple R-modules. Then:*

(1) λ is a hereditary pretorsion class, i.e., λ is closed under submodules, homomorphic images, and any direct sums;

(2) for any $N \in \lambda$, $N = \sum Rm$, where $m \in M^{(\mathbb{N})}$;

(3) the pullback and pushout of morphisms in λ belong to λ.

Corollary 3.3. *Let R be a semisimple ring and let M be a chain and faithful R-module and a subgenerator for all semisimple R-modules. If N is M-injective, then N is K-injective for all K λ-modules.*

Proof. If N is M-injective, then N is K-injective for any $K \in \sigma[M]$. However, according to Corollary 3.1, we get $\sigma[M] = \lambda$. Hence, N is K-injective for any $K \in \lambda$.

We now recall the following definition from Definition 2.6 in [10]:

Definition 3.1. *Let M and N be R-modules. We say that M rises to N and write $M \uparrow N$ if every M-injective module is N-injective.*

By using this definition and the properties of injectivity in $\sigma[M]$, we conclude that if $N \in \sigma[M]$, then $M \uparrow N$ but the converse is not necessary true. Theorem 2.8 in [10] gives a sufficient condition under which the converse assertion is true.

Corollary 3.4. *Let R be a semisimple ring and let M be a chain and faithful R-module and a subgenerator for all semisimple R-modules. For any module N such that $M \uparrow N$, N is M-injective if and only if N is K-injective for all $K \in \lambda$.*

Proof. The proof immediately follows from Corollary 3.3 and Theorem 2.8 of [10].

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