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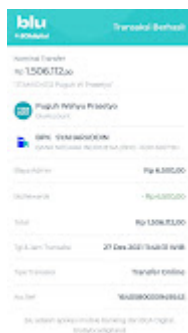Editor-Chief JTAM

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Brown-McCoy Radical in Restricted Graded Version

ABSTRACT

Some conjectures related to the radical theory of rings are still open. Hence, the research on the radical theory of rings is still being investigated by some prominent authors. On the other hand, some results on the radical theory of rings can be implemented in another branch or structure. In radical theor, it is interesting to bring some radical classes into graded versions. In this chance, we implement a qualitative method to conduct the research to bring the Brown-McCoy radical class to the restricted graded Brown-McCoy radical class. We start from some known facts on the Brown-McCoy radical class and furthermore, let G be a group, we explain the Brown-McCoy radical restricted with respect to the group G. The Brown-McCoy radical in restricted graded version will be denoted by $\mathcal{G}^G$ and we also give the fact by explaining what $\mathcal{G}^G(R)$ is as the final outcome of this paper.

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A. INTRODUCTION

In this paper, we use $I \lhd R$ to express I ideal of R. Let R be a ring and let a be a nonzero element of R. The element a is called a nilpotent element if $a^n = 0$ for some $n \in \mathbb{Z}^+$. If $\forall a \in R, a^n = 0$ for some $n \in \mathbb{Z}^+$, R is called a nil ring. Based on its history, the concept of radical class was observed by Köethe by investigating the property of the class $\mathcal{N} = \{R \mid \forall r \in R, \exists n \in \mathbb{Z}^+ \ni r^n = 0\}$ of rings. In other words, the class $\mathcal{N}$ consists of all nil rings (Gardner & Wiegandt, 2004). Köethe also discovered some facts on the class $\mathcal{N}$ of all nil rings. For all rings $R \in \mathcal{N}$ then $R/I \in \mathcal{N}, \forall I \lhd R$. Furthermore, the largest ideal I of a ring R which consists of nilpotent elements, is also the member of the class $\mathcal{N}$. Finally, if there exists ring R and $I \lhd R \ni I \in \mathcal{N}$ and $R/I \in \mathcal{N}$ implies $R \in \mathcal{N}$. These properties of the class $\mathcal{N}$ of all nil rings motivated Amitsur and Kurosh to define a radical class of rings. A class of rings γ is said to be radical if $A/I \in \gamma, \forall 0 \neq I$ proper ideal of A, for every ring $R, \gamma(R) = \Sigma\{I \lhd R \mid I \in \gamma\} \in \gamma$ and for every ring R, there exist an ideal I of R and $I, R/I \in \gamma$ implies $R \in \gamma$ (Gardner & Wiegandt, 2004). Directly, we can infer that the class $\mathcal{N}$ of all nil rings is a radical class. However, the class $\mathcal{N}_0 = \{R \text{ is a ring} \mid R^n = \{0\}$ for some $n \in \mathbb{Z}^+\}$ is not a radical class. In the development of radical theory, there are two types radical based on the construction. The lower radical and the upper radical. The lower radical $\mathcal{L}\mathcal{N}_0$ of the class $\mathcal{N}_0$ is being investigated by Baer, and it is denoted by β. Furthermore, since it follows from the fact that the radical class β is precisely the upper radical class of the class of all prime rings, the Baer radical class β is also called the prime radical class of rings. On the other hand, the class $\mathcal{L}$ of all locally nilpotent rings forms a radical class of rings and it is called the Levitzki

radical. The structure of Levitzki radical can be seen in (Gardner & Wiegandt, 2004), and the implementation of Levitzki radical in a skew polynomial ring can be seen in (Hong & Kim, 2019).

The nilpotent property of skew generalized power series ring has been discussed in (Ouyang & Liu, 2013). Moreover, a nilpotent derived from the implementation Jacobson radical class $\mathcal{J} = \{A | (A, \circ)$ forms a group, where $a \circ b = a + b - ab$ for every $a, b \in A\}$ of graded group ring is described in (Ilić-Georgijević, 2021). The definition of a graded ring will be given later in Definition 7 in this section. In fact, it follows from (Prasetyo & Melati, 2020) that the Jacobson radical $\mathcal{J}(A)$ of a ring A is two-sided brace. Some property of nilpotent group and the property of skew left brace of nilpotent type are described in (Smoktunowicz, 2018) and (Cedó et al., 2019).

On the other hand, $0 \neq J \lhd R, J$ is essential if $J \cap K \neq \{0\}, \forall 0 \neq K \lhd R$ and it will be denoted by $J \lhd\circ R$. Moreover, μ is special if μ consists of prime rings, for every ring $R \in \mu$ then every nonzero ideal I of R is also contained in μ, and for every essential ideal J of R such that $J \in \mu$ implies $R \in \mu$. An upper radical class $U(\mu)$ of a special class μ is special. It follows from the fact that β is precisely $U(\pi)$, where π is the class of all prime rings. Hence, β is special. Some properties related to special classes of rings and their generalization and their implementation in the development of the radical theory of rings and modules can be seen in (France-Jackson et al., 2015; Prasetyo et al., 2017, 2020; Prasetyo, Setyaningsih, et al., 2016; Prasetyo, Wijayanti, et al., 2016; Wahyuni et al., 2017).

Furthermore, the class of all simple rings with unity is denoted by $\mathcal{M}$. The upper radical $U(\mathcal{M})$ was being observed by Brown and McCoy, and it is called the Brown-McCoy radical class, and it is denoted by $\mathcal{G}$ (Gardner & Wiegandt, 2004). We provide some examples of simple rings with unity and their counterexamples.

Example 1. Consider the following concrete simple rings with unity
1. Every field is a simple ring with unity.
2. Let F be a field. Then the set $M_n(F)$ of all matrices of the size $n \times n$ over F forms a simple ring with unity

We give simple concrete rings which do not contain unity.
Example 2.
1. Let $M_\infty(R)$ be the ring of all infinite matrices which are row infinite over a ring R, that is, every matrix in $M_\infty(R)$ has a countably infinite number of rows, but almost all entries in each row are equal to 0. In the case, R is a field, then $M_\infty(R)$ is simple. Clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, the simple ring $M_\infty(R)$ does not contain the identity element.
2. The ideal $2\mathbb{Z}_4 = \{0,2\}$ of $\mathbb{Z}_4 = \{0,1,2,3\}$ is a simple ring, and it does not have unity.
   The Cayley tables of the addition and multiplication modulo 4 of $\mathbb{Z}_4$ respectively are shown in Table 1 and Table 2 below.

Table 1. Addition modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2. Multiplication modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	0	0	0

1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We further use the set $\mathbb{Z}_4$ of all integer numbers modulo 4 to construct a graded ring in Example 9.

In fact, for every special class μ of rings, μ is essentially closed. However, every class of prime rings does not necessarily to be essentially closed. We provide the following examples to make these conditions clear.

Example 3.

The π is special (Gardner & Wiegandt, 2004).

Example 4.

Let R be a prime ring. The prime ring R is called a $*-$ring if $R/I \in \beta$ for every nonzero proper ideal I of R. The definition of $*-$ring was introduced by Halina Korolczuk in 1981 (Prasetyo et al., 2017). Let $*$ be the symbol to express the class of all $*-$rings. However, $*$ is not closed under essential extension.

It follows from Example 4 that there is a class of prime rings, but it is not essentially closed. This condition motivated the existence of the definition of essential cover and essential closure.

Definition 5.

Let δ be any class of rings. The essential cover of δ is denoted by $\varepsilon(\delta)$, and it is defined as $\varepsilon(\delta) = \{A | \exists B \triangleleft\circ A \ni B \in \delta\}$. Moreover, if δ is closed under essential extension, then $\varepsilon(\delta) = \delta$. Furthermore, the set $\delta_k = \bigcup_{t=0}^{\infty} \delta^t = 0$ is the essential closure of δ, where $\delta^{(0)} = \delta$ and $\delta^{(t+1)} = \varepsilon(\delta^{(t)})$.

In general, the essential closure μ_k of a special class of rings μ is μ itself since $\mu^{(t)} = \mu$ for every $t \in \{0,1,2,\dots\}$. However, in the case of nonspecial class of ring, the essential closure δ_k of δ strictly contains δ. We provide the following example.

Example 6.

It follows from Example 4 that the class $*$ is not special. The essential closure of $*_k$ of $*$ strictly contains $*$.

A graded ring is one of the kinds of rings such that its structure is being investigated by prominent authors. We give highlight some research outcomes related to the existence of graded rings. In the point of view of an epsilon category, the multiplicity of graded algebras and epsilon-strongly groupoid graded can be found in (Das, 2021) and (Nystedt et al., 2020), respectively. The studies on graded ring related to Leavitt path algebra can be seen in (R. Hazrat, 2014; Roozbeh Hazrat et al., 2018; Lännström, 2020; Vaš, 2020a, 2020b). Furthermore, the studies on graded ring related to the specific structure of rings and modules, namely weakly prime ring, non-commutative rings, prime spectrum, unique factorization rings, positively graded rings, simple rings, $S-$Noetherian ring, and Dedekind rings can be accessed in (Abu-Dawwasb, 2018; Al-Zoubi & Jaradat, 2018; Alshehry & Abu-Dawwasb, 2021; Çeken & Alkan, 2015; Ernanto et al., 2020; Kim & Lim, 2020; Nystedt & Öinert, 2020; Wahyuni et al., 2020; Wijayanti et al., 2020) respectively. Thus, it is interesting to investigate some further properties and structures related to graded rings. On the other hand, the purpose of this research is to

determine what $\mathcal{G}^G(R)$ is. Finally, in the following definition, we provide the definition a graded ring.

Definition 7.
let G be a monoid. A ring R is a $G-$graded ring if $R = \bigoplus_{g \in G} R_g$ where the set $\{R_g | g \in G\}$ is the collection of additive subgroups of $R \ni R_g R_h \subseteq R_{gh} \forall g, h \in G$ (Kim & Lim, 2020).

In order to make the reader be clear in understanding the concept of a graded ring, we provide the following examples.

Example 8.
Let $R(x)$ be the set of all infinite sequences $(a_0, a_1, a_2, a_3, \dots)$ where $a_i \in R$ and R is a ring for every $i \in \{0,1,2,3, \dots\}$ and where there exists $n \in G = \{0,1,2,3 \dots\}$ such that for all integers $k \geq n, a_k = 0$. Directly we can infer that the polynomial ring $R(x) = \bigoplus_{g \in G} R_g$ where $R_0 = (a_0, 0,0, \dots), R_1 = (0, a_1, 0,0, \dots), \dots R_g = (0,0, \dots, a_g, 0,0, \dots)$ such that $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$. The polynomial $R(x)$ is graded, and it is graded by its degree. Therefore, for every ring R, the graded ring naturally exists.

We also provide another example of a graded ring as follows.

Example 9.
Consider the set $\mathbb{Z}_4 = \{0,1,2,3\}$ and let A be any ring. Now the set $M_{2 \times 2}(A)$ is the set of all 2×2 matrices over ring A. Define $A_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We therefore have $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ and $A_g A_h \subseteq A_{gh}$ for every $g, h \in Z_4$. Thus, we can infer that A is a $\mathbb{Z}_4-$graded ring.

Some further and nontrivial examples of graded rings can be accessed in (Alshehry & Abu-Dawwasb, 2021) and (Pratibha et al., 2017). Moreover, the monoid G can be strictly replaced by any arbitrary group. Let R and S be graded rings with respect to G, the symbol R^u will denote the underlying ungraded ring. A ring homomorphism f which maps R to S is called a graded homomorphism of degree (h, k) if $f(R_g) \subseteq S_{hgk}, \forall g \in G$. The existence of this graded homomorphism motivated the existence of the definition of graded radical. For simply, the definition of graded radical class can be seen in Definition 2 in (Fang & Stewart, 1992), which is similar to the definition of radical class in the graded version.

Furthermore, we shall follow the construction of the restricted graded radical introduced by Hongjin Fang and Patrick Stewart in their paper (Fang & Stewart, 1992). Let γ be radical, $\gamma^G = \{R | R \text{ is a } G-\text{graded ring and } A^u \in \gamma\}$. Moreover, for further consideration, it will be called a graded radical γ^G of γ.

Some properties of graded radical related to the normality and specialty of the graded radical can be seen in (Fang & Stewart, 1992). In this paper, we define $\mathcal{G}^G$ and determine what $\mathcal{G}^G(R)$ is.

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B. METHODS

This research is conducted using a qualitative method derived from facts and known concepts from a literature study. To gain some properties of radical of rings, we follow some concepts and radical construction (Gardner & Wiegandt, 2004). There are lower radical class and upper radical class. In this part, we focus on the upper radical class construction. We post

the known result in the early part of Section C. We start with the structure of what the Brown-McCoy radical of a ring actually is and prove the property completely.

In the development of graded radicals, the graded version of the radical class of rings has been being investigated. Gardner and Plant, in their paper (Gardner & Plant, 2009), investigated and compared the Jacobson radical and graded Jacobson radical. The homogeneity radicals defined by the nilpotency of a graded semigroup ring are described in (Hong et al., 2018). Some further fundamental structures of graded radical of rings can also be studied from (Lee & Puczylowski, 2014), (Hong et al., 2014) and (Mazurek et al., 2015). However, we shall follow the concept of restricted graded radical which was introduced by (Fang & Stewart, 1992) to construct the restricted graded Brown-McCoy radical. Finally, in virtue of the construction of the restricted graded Brown-McCoy radical; we can determine the structure of the restricted graded Brown-McCoy radical.

C. RESULT AND DISCUSSION

We separate this part into two subsections. In the first part, we describe what the Brown-McCoy of a ring actually is. Moreover, in the second part, we describe the construction of restricted graded Brown-McCoy radical and give some of its properties.

1. Brown-McCoy Radical

We start this part with this proposition as a reminder and we give complete proof.

Proposition 10. (Gardner & Wiegandt, 2004)

Let A be any ring, $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$

Proof.

In fact, $\mathcal{G} = \mathcal{U}(\mathcal{M}) = \{A |$ there is no $A/I \in \mathcal{M}$ for every I ideal of $A\}$y. Now let $A \in \mathcal{M}$. Suppose I and K be ideals of A such that $IK = \{0\}$. Since the ring A is a simple ring, in the case of $K = A$ implies $I = \{0\}$ or in the case of $I = A$ implies $K = \{0\}$. Hence, $\{0\}$ is a prime ideal of A. Thus, A is prime. So, we may deduce that $\mathcal{M}$ consists of prime rings. Furthermore $\mathcal{M}$ is hereditary, and it is essentially closed. This gives $\mathcal{M}$ is special. Thus $\mathcal{G}$ is hereditary. Furthermore, since $\mathcal{M}$ is essentially closed, the essential cover $\mathcal{EM}$ of $\mathcal{M}$ coincide with $\mathcal{M}$. It follows from Theorem 3.7.2 in (Gardner & Wiegandt, 2004) that $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$. ∎

2. Graded Brown-McCoy Radical

Let G be any group. It follows from the construction of restricted $G-$graded radical described in (Fang & Stewart, 1992) that a restricted $G-$graded Brown-McCoy radical class can be defined as $\mathcal{G}^G = \{A$ is a $G-$graded ring $| A^u \in \mathcal{G}\}$, where A^u is underlying ungraded of the ring A. For further consideration, we call the class of rings $\mathcal{G}^G$ by graded Brown-McCoy radical. In the next theorem, we will show that the graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Theorem 11.

Let G be any group. The restricted $G-$graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Proof.

It can be directly inferred by the hereditaries of the Brown-McCoy radical $\mathcal{G}$ and Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G$ is hereditary. However, we will provide the proof in detail for the reader. Now let $A \in \mathcal{G}^G$ and let I be any nonzero homogenous proper ideal of A. Then $A = \bigoplus_{g \in G} A_g$ and $I = \bigoplus_{h \in G} I \cap A_h$. Hence, I is a $G-$graded ring. It is clear that I^u is a ideal of $A^u \in \mathcal{G}$. Since $\mathcal{G}$ is hereditary, $I^u \in \mathcal{G}$. So, we can deduce that $I \in \mathcal{G}^G$. Thus, $\mathcal{G}^G$ is hereditary. ∎

The hereditaries of $\mathcal{G}^G$ explained in Theorem 11 implies the following property.

Theorem 12.
Let G be any group, $\mathcal{G}^G(A)$ is the intersection of all $G-$graded ideals I of A such that A/I is $G-$graded simple ring with unity, where A is a $G-$graded ring.

Proof.
Let A be any $G-$graded ring. In virtue of Theorem 2, $\mathcal{G}^G$ is hereditary. Furthermore, since $\mathcal{G}^G$ is hereditary and it follows from Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G(A) = (\mathcal{G}(A))_G$, where $(\mathcal{G}(A))_G = \bigoplus_{g \in G} \{\mathcal{G}(A) \cap A_g | g \in G\}$. Now let $\{I_\lambda\}, \lambda \in \Lambda$, where Λ is index, be the collection of all ideals of A such that $A/I_\lambda \in \mathcal{M}$. Define $I_\lambda^G = \bigoplus_{g \in G} I_\lambda \cap A_g$ for every $\lambda \in \Lambda$. It is clear that I_λ^G is a $G-$graded ideal of A such that I_λ^G is simple $G-$graded with unity. Now the intersection of all $\{I_\lambda\}, \lambda \in \Lambda$ is

$$
\begin{aligned}
\cap_{\lambda \in \Lambda} I_\lambda^G \quad &= \cap_{\lambda \in \Lambda} \left\{ \bigoplus_{g \in G} I_\lambda \cap A_g \right\} \\
&= \bigoplus_{g \in G} \left\{ (\cap_{\lambda \in \Lambda} I_\lambda) \cap A_g \right\} \\
\cap_{\lambda \in \Lambda} I_\lambda^G \quad &= \bigoplus_{g \in G} \left\{ \mathcal{G}(A) \cap A_g | g \in G \right\}
\end{aligned} \tag{1}
$$

On the other hand,

$$
\mathcal{G}^G(A) = (\mathcal{G}(A))_G \tag{2}
$$

It follows from equations (1) and (2) that $(\mathcal{G}(A))_G$ is precisely the intersection of all $G-$graded ideals I of A such that A/I is $G-$graded simple ring with unity which completes the proof. ∎

D. CONCLUSION AND SUGGESTIONS
It follows from Theorem 12 that for a fixed group G, and let A be ring, $\mathcal{G}^G(A) = \cap \{I_\lambda | A/I_\lambda$ is a $G-$graded which is a member of $\mathcal{M}\}$. The research can be continued to investigate the restricted $G-$graded for Levitzki radical, Thierrin radical, anti simple radical, and Behrens radical.

> **Commented [M3]:** It is best to conclude that it will answer from the purpose of the research.

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Brown-McCoy Radical in Restricted Graded Version

ABSTRACT

Some conjectures related to the radical theory of rings are still open. Hence, the research on the radical theory of rings is still being investigated by some prominent authors. On the other hand, some results on the radical theory of rings can be implemented in another branch or structure. In radical theor, it is interesting to bring some radical classes into graded versions. In this chance, we implement a qualitative method to conduct the research to bring the Brown-McCoy radical class to the restricted graded Brown-McCoy radical class. We start from some known facts on the Brown-McCoy radical class and furthermore, let G be a group, we explain the Brown-McCoy radical restricted with respect to the group G. The Brown-McCoy radical in restricted graded version will be denoted by $\mathcal{G}^G$ and we also give the fact by explaining what $\mathcal{G}^G(R)$ is as the final outcome of this paper.

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A. INTRODUCTION

In this paper, we use $I \lhd R$ to express I ideal of R. Let R be a ring and let a be a nonzero element of R. The element a is called a nilpotent element if $a^n = 0$ for some $n \in \mathbb{Z}^+$. If $\forall a \in R, a^n = 0$ for some $n \in \mathbb{Z}^+$, R is called a nil ring. Based on its history, the concept of radical class was observed by Köethe by investigating the property of the class $\mathcal{N} = \{R \mid \forall r \in R, \exists n \in \mathbb{Z}^+ \ni r^n = 0\}$ of rings. In other words, the class $\mathcal{N}$ consists of all nil rings (Gardner & Wiegandt, 2004). Köethe also discovered some facts on the class $\mathcal{N}$ of all nil rings. For all rings $R \in \mathcal{N}$ then $R/I \in \mathcal{N}, \forall I \lhd R$. Furthermore, the largest ideal I of a ring R which consists of nilpotent elements, is also the member of the class $\mathcal{N}$. Finally, if there exists ring R and $I \lhd R \ni I \in \mathcal{N}$ and $R/I \in \mathcal{N}$ implies $R \in \mathcal{N}$. These properties of the class $\mathcal{N}$ of all nil rings motivated Amitsur and Kurosh to define a radical class of rings. A class of rings γ is said to be radical if $A/I \in \gamma, \forall 0 \neq I$ proper ideal of A, for every ring $R, \gamma(R) = \Sigma\{I \lhd R | I \in \gamma\} \in \gamma$ and for every ring R, there exist an ideal I of R and $I, R/I \in \gamma$ implies $R \in \gamma$ (Gardner & Wiegandt, 2004). Directly, we can infer that the class $\mathcal{N}$ of all nil rings is a radical class. However, the class $\mathcal{N}_0 = \{R$ is a ring $| R^n = \{0\}$ for some $n \in \mathbb{Z}^+\}$ is not a radical class. In the development of radical theory, there are two types radical based on the construction. The lower radical and the upper radical. The lower radical $\mathcal{L}\mathcal{N}_0$ of the class $\mathcal{N}_0$ is being investigated by Baer, and it is denoted by β. Furthermore, since it follows from the fact that the radical class β is precisely the upper radical class of the class of all prime rings, the Baer radical class β is also called the prime radical class of rings. On the other hand, the class $\mathcal{L}$ of all locally nilpotent rings forms a

radical class of rings and it is called the Levitzki radical. The structure of Levitzki radical can be seen in (Gardner & Wiegandt, 2004), and the implementation of Levitzki radical in a skew polynomial ring can be seen in (Hong & Kim, 2019).

The nilpotent property of skew generalized power series ring has been discussed in (Ouyang & Liu, 2013). Moreover, a nilpotent derived from the implementation Jacobson radical class $\mathcal{J} = \{A|(A,\circ)$ forms a group, where $a \circ b = a + b - ab$ for every $a, b \in A\}$ of graded group ring is described in (Ilić-Georgijević, 2021). The definition of a graded ring will be given later in Definition 7 in this section. In fact, it follows from (Prasetyo & Melati, 2020) that the Jacobson radical $\mathcal{J}(A)$ of a ring A is two-sided brace. Some property of nilpotent group and the property of skew left brace of nilpotent type are described in (Smoktunowicz, 2018) and (Cedó et al., 2019).

On the other hand, $0 \neq J \lhd R, J$ is essential if $J \cap K \neq \{0\}, \forall 0 \neq K \lhd R$ and it will be denoted by $J \lhd\circ R$. Moreover, μ is special if μ consists of prime rings, for every ring $R \in \mu$ then every nonzero ideal I of R is also contained in μ, and for every essential ideal J of R such that $J \in \mu$ implies $R \in \mu$. An upper radical class $U(\mu)$ of a special class μ is special. It follows from the fact that β is precisely $U(\pi)$, where π is the class of all prime rings. Hence, β is special. Some properties related to special classes of rings and their generalization and their implementation in the development of the radical theory of rings and modules can be seen in (France-Jackson et al., 2015; Prasetyo et al., 2017, 2020; Prasetyo, Setyaningsih, et al., 2016; Prasetyo, Wijayanti, et al., 2016; Wahyuni et al., 2017).

Furthermore, the class of all simple rings with unity is denoted by $\mathcal{M}$. The upper radical $U(\mathcal{M})$ was being observed by Brown and McCoy, and it is called the Brown-McCoy radical class, and it is denoted by $\mathcal{G}$ (Gardner & Wiegandt, 2004). We provide some examples of simple rings with unity and their counterexamples.

Example 1. Consider the following concrete simple rings with unity
1. Every field is a simple ring with unity.
2. Let F be a field. Then the set $M_n(F)$ of all matrices of the size $n \times n$ over F forms a simple ring with unity

We give simple concrete rings which do not contain unity.
Example 2.
1. Let $M_\infty(R)$ be the ring of all infinite matrices which are row infinite over a ring R, that is, every matrix in $M_\infty(R)$ has a countably infinite number of rows, but almost all entries in each row are equal to 0. In the case, R is a field, then $M_\infty(R)$ is simple. Clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, the simple ring $M_\infty(R)$ does not contain the identity element.
2. The ideal $2\mathbb{Z}_4 = \{0,2\}$ of $\mathbb{Z}_4 = \{0,1,2,3\}$ is a simple ring, and it does not have unity.
   The Cayley tables of the addition and multiplication modulo 4 of $\mathbb{Z}_4$ respectively are shown in Table 1 and Table 2 below.

Table 1. Addition modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2. Multiplication modulo 4 of $\mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We further use the set $\mathbb{Z}_4$ of all integer numbers modulo 4 to construct a graded ring in Example 9.

In fact, for every special class μ of rings, μ is essentially closed. However, every class of prime rings does not necessarily to be essentially closed. We provide the following examples to make these conditions clear.

Example 3.

The π is special (Gardner & Wiegandt, 2004).

Example 4.

Let R be a prime ring. The prime ring R is called a $\ast -$ring if $R/I \in \beta$ for every nonzero proper ideal I of R. The definition of $\ast -$ring was introduced by Halina Korolczuk in 1981 (Prasetyo et al., 2017). Let $\ast$ be the symbol to express the class of all $\ast -$rings. However, $\ast$ is not closed under essential extension.

It follows from Example 4 that there is a class of prime rings, but it is not essentially closed. This condition motivated the existence of the definition of essential cover and essential closure.

Definition 5.

Let δ be any class of rings. The essential cover of δ is denoted by $\varepsilon(\delta)$, and it is defined as $\varepsilon(\delta) = \{A | \exists B \lhd \circ A \ni B \in \delta\}$. Moreover, if δ is closed under essential extension, then $\varepsilon(\delta) = \delta$. Furthermore, the set $\delta_k = \bigcup_{t=0}^{\infty} \delta^t = 0$ is the essential closure of δ, where $\delta^{(0)} = \delta$ and $\delta^{(t+1)} = \varepsilon(\delta^{(t)})$.

In general, the essential closure μ_k of a special class of rings μ is μ itself since $\mu^{(t)} = \mu$ for every $t \in \{0,1,2,\dots\}$. However, in the case of nonspecial class of ring, the essential closure δ_k of δ strictly contains δ. We provide the following example.

Example 6.

It follows from Example 4 that the class $\ast$ is not special. The essential closure of $\ast_k$ of $\ast$ strictly contains $\ast$.

A graded ring is one of the kinds of rings such that its structure is being investigated by prominent authors. We give highlight some research outcomes related to the existence of graded rings. In the point of view of an epsilon category, the multiplicity of graded algebras and epsilon-strongly groupoid graded can be found in (Das, 2021) and (Nystedt et al., 2020), respectively. The studies on graded ring related to Leavitt path algebra can be seen in (R. Hazrat, 2014; Roozbeh Hazrat et al., 2018; Lännström, 2020; Vaš, 2020a, 2020b). Furthermore, the studies on graded ring related to the specific structure of rings and modules, namely weakly prime ring, non-commutative rings, prime spectrum, unique factorization rings, positively graded rings, simple rings, $S -$Noetherian ring, and Dedekind rings can be accessed in (Abu-Dawwasb, 2018; Al-Zoubi & Jaradat, 2018; Alshehry & Abu-Dawwasb, 2021; Çeken & Alkan, 2015; Ernanto et al., 2020; Kim & Lim, 2020; Nystedt & Öinert, 2020; Wahyuni

et al., 2020; Wijayanti et al., 2020) respectively. Thus, it is interesting to investigate some further properties and structures related to graded rings. On the other hand, the purpose of this research is to determine what $\mathcal{G}^G(R)$ is. Finally, in the following definition, we provide the definition a graded ring.

Definition 7.
let G be a monoid. A ring R is a $G-$graded ring if $R = \bigoplus_{g \in G} R_g$ where the set $\{R_g | g \in G\}$ is the collection of additive subgroups of $R \ni R_g R_h \subseteq R_{gh} \forall g, h \in G$ (Kim & Lim, 2020).

In order to make the reader be clear in understanding the concept of a graded ring, we provide the following examples.

Example 8.
Let $R(x)$ be the set of all infinite sequences $(a_0, a_1, a_2, a_3, \dots)$ where $a_i \in R$ and R is a ring for every $i \in \{0,1,2,3,\dots\}$ and where there exists $n \in G = \{0,1,2,3 \dots\}$ such that for all integers $k \geq n, a_k = 0$. Directly we can infer that the polynomial ring $R(x) = \bigoplus_{g \in G} R_g$ where $R_0 = (a_0, 0, 0, \dots), R_1 = (0, a_1, 0, 0, \dots), \dots R_g = (0, 0, \dots, a_g, 0, 0, \dots)$ such that $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$. The polynomial $R(x)$ is graded, and it is graded by its degree. Therefore, for every ring R, the graded ring naturally exists.

We also provide another example of a graded ring as follows.

Example 9.
Consider the set $\mathbb{Z}_4 = \{0,1,2,3\}$ and let A be any ring. Now the set $M_{2\times2}(A)$ is the set of all 2×2 matrices over ring A. Define $A_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We therefore have $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ and $A_g A_h \subseteq A_{gh}$ for every $g, h \in Z_4$. Thus, we can infer that A is a $\mathbb{Z}_4-$graded ring.

Some further and nontrivial examples of graded rings can be accessed in (Alshehry & Abu-Dawwasb, 2021) and (Pratibha et al., 2017). Moreover, the monoid G can be strictly replaced by any arbitrary group. Let R and S be graded rings with respect to G, the symbol R^u will denote the underlying ungraded ring. A ring homomorphism f which maps R to S is called a graded homomorphism of degree (h, k) if $f(R_g) \subseteq S_{hgk}, \forall g \in G$. The existence of this graded homomorphism motivated the existence of the definition of graded radical. For simply, the definition of graded radical class can be seen in Definition 2 in (Fang & Stewart, 1992), which is similar to the definition of radical class in the graded version.

Furthermore, we shall follow the construction of the restricted graded radical introduced by Hongjin Fang and Patrick Stewart in their paper (Fang & Stewart, 1992). Let γ be radical, $\gamma^G = \{R | R \text{ is a } G-\text{graded ring and } A^u \in \gamma\}$. Moreover, for further consideration, it will be called a graded radical γ^G of γ.

Some properties of graded radical related to the normality and specialty of the graded radical can be seen in (Fang & Stewart, 1992). In this paper, we define $\mathcal{G}^G$ and determine what $\mathcal{G}^G(R)$ is.

B. METHODS

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This research is conducted using a qualitative method derived from facts and known concepts from a literature study. To gain some properties of radical of rings, we follow some concepts and radical construction (Gardner & Wiegandt, 2004). There are lower radical class and upper radical class. In this part, we focus on the upper radical class construction. We post the known result in the early part of Section C. We start with the structure of what the Brown-McCoy radical of a ring actually is and prove the property completely.

In the development of graded radicals, the graded version of the radical class of rings has been being investigated. Gardner and Plant, in their paper (Gardner & Plant, 2009), investigated and compared the Jacobson radical and graded Jacobson radical. The homogeneity radicals defined by the nilpotency of a graded semigroup ring are described in (Hong et al., 2018). Some further fundamental structures of graded radical of rings can also be studied from (Lee & Puczylowski, 2014), (Hong et al., 2014) and (Mazurek et al., 2015). However, we shall follow the concept of restricted graded radical which was introduced by (Fang & Stewart, 1992) to construct the restricted graded Brown-McCoy radical. Finally, in virtue of the construction of the restricted graded Brown-McCoy radical; we can determine the structure of the restricted graded Brown-McCoy radical.

C. RESULT AND DISCUSSION

We separate this part into two subsections. In the first part, we describe what the Brown-McCoy of a ring actually is. Moreover, in the second part, we describe the construction of restricted graded Brown-McCoy radical and give some of its properties.

1. Brown-McCoy Radical

We start this part with this proposition as a reminder and we give complete proof.

Proposition 10. (Gardner & Wiegandt, 2004)

Let A be any ring, $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$

Proof.

In fact, $\mathcal{G} = \mathcal{U}(\mathcal{M}) = \{A | \text{ there is no } A/I \in \mathcal{M} \text{ for every } I \text{ ideal of } A\}y$. Now let $A \in \mathcal{M}$. Suppose I and K be ideals of A such that $IK = \{0\}$. Since the ring A is a simple ring, in the case of $K = A$ implies $I = \{0\}$ or in the case of $I = A$ implies $K = \{0\}$. Hence, $\{0\}$ is a prime ideal of A. Thus, A is prime. So, we may deduce that $\mathcal{M}$ consists of prime rings. Furthermore $\mathcal{M}$ is hereditary, and it is essentially closed. This gives $\mathcal{M}$ is special. Thus $\mathcal{G}$ is hereditary. Furthermore, since $\mathcal{M}$ is essentially closed, the essential cover $\mathcal{EM}$ of $\mathcal{M}$ coincide with $\mathcal{M}$. It follows from Theorem 3.7.2 in (Gardner & Wiegandt, 2004) that $\mathcal{G}(A) = \cap \{I_\lambda | A/I_\lambda \in \mathcal{M}\}$. ∎

2. Graded Brown-McCoy Radical

Let G be any group. It follows from the construction of restricted $G-$graded radical described in (Fang & Stewart, 1992) that a restricted $G-$graded Brown-McCoy radical class can be defined as $\mathcal{G}^G = \{A \text{ is a } G-\text{graded ring} | A^u \in \mathcal{G}\}$, where A^u is underlying ungraded of the ring A. For further consideration, we call the class of rings $\mathcal{G}^G$ by graded Brown-McCoy radical. In the next theorem, we will show that the graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Theorem 11.

Let G be any group. The restricted $G-$graded Brown-McCoy radical $\mathcal{G}^G$ is hereditary.

Proof.

It can be directly inferred by the hereditaries of the Brown-McCoy radical $\mathcal{G}$ and Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G$ is hereditary. However, we will provide the proof in detail for

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the reader. Now let $A \in \mathcal{G}^G$ and let I be any nonzero homogenous proper ideal of A. Then $A = \bigoplus_{g \in G} A_g$ and $I = \bigoplus_{h \in G} I \cap A_h$. Hence, I is a $G-$graded ring. It is clear that I^u is a ideal of $A^u \in \mathcal{G}$. Since $\mathcal{G}$ is hereditary, $I^u \in \mathcal{G}$. So, we can deduce that $I \in \mathcal{G}^G$. Thus, $\mathcal{G}^G$ is hereditary. ∎

The hereditaries of $\mathcal{G}^G$ explained in Theorem 11 implies the following property.

Theorem 12.
Let G be any group, $\mathcal{G}^G(A)$ is the intersection of all $G-$graded ideals I of A such that A/I is $G-$graded simple ring with unity, where A is a $G-$graded ring.
Proof.
Let A be any $G-$graded ring. In virtue of Theorem 2, $\mathcal{G}^G$ is hereditary. Furthermore, since $\mathcal{G}^G$ is hereditary and it follows from Proposition 2 in (Fang & Stewart, 1992) that $\mathcal{G}^G(A) = (\mathcal{G}(A))_G$, where $(\mathcal{G}(A))_G = \bigoplus_{g \in G} \{\mathcal{G}(A) \cap A_g | g \in G\}$. Now let $\{I_\lambda\}, \lambda \in \Lambda$, where Λ is index, be the collection of all ideals of A such that $A/I_\lambda \in \mathcal{M}$. Define $I_\lambda^G = \bigoplus_{g \in G} I_\lambda \cap A_g$ for every $\lambda \in \Lambda$. It is clear that I_λ^G is a $G-$graded ideal of A such that I_λ^G is simple $G-$graded with unity. Now the intersection of all $\{I_\lambda\}, \lambda \in \Lambda$ is

$$\begin{aligned} \cap_{\lambda \in \Lambda} I_\lambda^G &= \cap_{\lambda \in \Lambda} \left\{ \bigoplus_{g \in G} I_\lambda \cap A_g \right\} \\ &= \bigoplus_{g \in G} \left\{ (\cap_{\lambda \in \Lambda} I_\lambda) \cap A_g \right\} \\ \cap_{\lambda \in \Lambda} I_\lambda^G &= \bigoplus_{g \in G} \left\{ \mathcal{G}(A) \cap A_g | g \in G \right\} \end{aligned} \tag{1}$$

On the other hand,

$$\mathcal{G}^G(A) = (\mathcal{G}(A))_G \tag{2}$$

It follows from equations (1) and (2) that $(\mathcal{G}(A))_G$ is precisely the intersection of all $G-$graded ideals I of A such that A/I is $G-$graded simple ring with unity which completes the proof. ∎

D. **CONCLUSION** AND SUGGESTIONS

It follows from Theorem 12 that for a fixed group G, and let A be ring, $\mathcal{G}^G(A) = \cap \{I_\lambda | A/I_\lambda$ is a $G-$graded which is a member of $\mathcal{M}\}$. The research can be continued to investigate the restricted $G-$graded for Levitzki radical, Thierrin radical, anti simple radical, and Behrens radical.

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