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On the graded $*$ -rings

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Abstract

A prime ring A is called a $*$ -ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A , where β is the prime radical. Gardner in 1988 asked whether the upper radical $\mathcal{U}(*_k)$ of the essential closure $*_k$ of the class of all $*$ -rings coincide with the prime radical β . Until now, this problem remains open. In this paper, we construct a graded $*$ -ring that motivates a further research to bring the Gardner problem into a graded version.

1 Introduction

In this paper, R (respectively, A) will denote a ring which has identity (respectively, a ring which is not necessary to have identity). A nonzero ideal of J of a ring R

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is essential if $I \cap J \neq 0$ for every nonzero ideal I of R and it will be denoted by $I \triangleleft R$.

A class γ of rings is called a radical class in the sense of Amitsur and Kurosh if γ satisfies the following property [7]:

1. $A \in \gamma \Rightarrow \forall A \rightarrow B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma$.
2. A is arbitrary ring and $\forall A \rightarrow B \neq 0$, there exists an ideal C of B such that $0 \neq C \in \gamma \Rightarrow A \in \gamma$.

Example 1.1. The following examples are radical class of rings:

1. The class $\mathcal{N} = \{A \mid \forall a \in A, \text{ then there exists a positive integer } n \text{ such that } a^n = 0\}$ of all nil rings is a radical class.
2. The class $\mathcal{J} = \{A \text{ is a ring } \mid (A, \circ) \text{ forms a group, where } a \circ b = a + b - ab, \forall a, b \in A\}$ forms a radical class.
3. Let π be the class of all prime rings. The class $\mathcal{U}(\pi) = \{A \mid A \text{ has no nonzero homomorphic image is } \pi\}$ is a radical class. It is famously named the prime radical class and it is denoted by β .

For any radical class γ . The symbol $\gamma(A)$ represents the largest ideal of a ring A which belongs to γ . A prime ring A is called a $*$ -ring if $\beta(A/I) = A/I$ for every nonzero ideal proper I of A , where β is the prime radical. The class of all $*$ -ring is denoted by $*$ [2]. The definition of $*$ -ring was introduced in [11] and some further properties of $*$ -rings and their implementation in radical theory can be seen in [3, 4, 5, 6].

Example 1.2. Consider the following examples of $*$ -rings.

1. Every field is a $*$ -ring.
2. The set $J = \{j \in \mathbb{Q} \mid j \text{ has even numerator and odd denominator}\}$ forms a $*$ -ring.
3. Let W be a simple idempotent ring of characteristic 0, but with no unity. The ring W is also a $*$ -ring.
4. Every simple ring with unity is a $*$ -ring.

On the other hand, we have also the following conditions (S1 and S2) which motivate the existence of the definition of semisimple class of rings

1. (S1) If $A \in \sigma$, then for every nonzero $B \triangleleft A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$.
2. (S2) If A is a ring of the universal class \mathbb{A} and for every nonzero $B \triangleleft A$ there exists a nonzero homomorphic image C of B such that $C \in \sigma$, then A is in σ .

A class σ of rings which satisfies the condition S1 is called a regular class of rings. Furthermore, if the class σ of rings satisfies S1 and S2, σ is called a semisimple class of rings [7].

The relationship between radical class and semisimple class is described in the Proposition 1.1 below.

Proposition 1.1. [7] *Let γ be a radical class of rings. The class $S\gamma = \{A \text{ is a ring} \mid \gamma(A) = 0\}$ is the semisimple class of γ .*

Proof. Please see the Proof of Proposition 2.3.2 in [7]. □

Furthermore, for any regular class ϱ of rings. We have the following property:

Theorem 1.1. [7] *If ϱ is a regular class of rings, then the class $\mathcal{U}(\varrho) = \{A \mid A \text{ has no nonzero homomorphic image in } \varrho\}$ is a radical class, $\mathcal{U}(\varrho) \cap \varrho = \{0\}$ and $\mathcal{U}(\varrho)$ is the largest radical having zero intersection with ϱ .*

Proof. Please see the proof of Theorem 2.2.3 in [7]. □

A class μ of prime rings is called a special class of rings if μ satisfies the following properties [7]:

1. $A \in \mu \Rightarrow I \in \mu$ for every ideal I of A .
2. for every essential ideal J of R such that $I \in \mu$ implies $R \in \mu$.

Example 1.3. *The class π of all prime rings is a special class. Moreover, the upper radical $\mathcal{U}(\pi)$ is precisely the prime radical β .*

In 1988, Gardner in his paper [8] asked whether the prime radical β coincides with the upper radical $\mathcal{U}(*_k)$, where $\mathcal{U}(*_k)$ will be explained more detail in the Section 2. However, this question remains open in general. Hence, it is important to investigate this problem at least on a specific condition. In this research, we give an insight of the problem specifically on the restricted graded version for radical class of rings.

2 The structure of $'*$

We start this section revisiting the properties and the structure of $*$.

Remark 2.1. [2] *The class $*$ is neither a radical class nor a semisimple class.*

Consider the following counter examples.

Example 2.1. *Let W be a simple idempotent ring of characteristic 0, but with no unity. It is clear that the ring J in the Example 1.2 is a $*$ -ring. The ring W is also a $*$ -ring. Define $R = \{(j, w) | j \in J, w \in W\}$ with $(j, w) + (k, x) = (j+k, w+x)$ and $(j, w)(k, x) = (jk, jx + kw + wx)$ for all $j, k \in J$ and $w, x \in W$. Then, R is a subdirectly irreducible ring with the heart $H(R) \cong W$ such that $R/H(R) \cong J$ and any Noetherian homomorphic image of R (other than the identical one and $R/H(R)$) is nilpotent. Therefore R is not a $*$ -ring in spite of the fact that both $R/H(R) \cong J$ and $H(R)$ are a $*$ -rings.*

Remark 2.2. *The existence of a simple idempotent ring of characteristic 0, but with no unity can be seen in [10] and [14].*

Example 2.2. *Let $M_\infty(R)$ be the ring of all infinite matrices which has infinite row over a ring R . In the other words, every matrix in $M_\infty(R)$ has countably infinite number of rows but almost all entries in each row are equal to 0. If R is a simple ring, then so $M_\infty(R)$ and, clearly, the center of $M_\infty(R)$ is $\{0\}$. Therefore, $M_\infty(R)$ does not contain the identity element. So, in particular, if R is any simple ring with characteristic 0, then so is $M_\infty(R)$.*

Remark 2.3. [2] *The class $*$ is not essentially closed. Therefore, the class $*$ is not a special class.*

Example 2.3. *Example 2.1 shows that the ring $R = \{(j, w) | j \in J, w \in W\}$, where $J = \{j \in \mathbb{Q} | j \text{ has even numerator and odd denominator}\}$ and W is a simple idempotent ring of characteristic 0 without unity, is a subdirectly irreducible. The heart $H(R)$ of R is $W' \cong W$. So, we may deduce that the ideal W' is an essential ideal of R and R is an essential extension of W' . In fact, $W' \cong W$ is a $*$ -ring. However, the ring R is not a $*$ -ring by Example 2.1. Therefore, the class $*$ is not a special class.*

Let δ be any class of rings which is not essentially closed. The class $\varepsilon(\delta) = \{A | \exists B \in \delta \text{ such that } B \triangleleft \circ A\}$ is the essential cover of δ , where $B \triangleleft \circ A$ expresses

B is the essential ideal of A . The definition of an essential ideal is previously described in the beginning Section 1. But, the class $\varepsilon(\delta)$ is not generally closed under essential extensions. This motivates the existence of essential closure of δ .

Definition 2.1. [9] Let $\delta^{(0)} = \delta$ and $\delta^{(i+1)} = \varepsilon(\delta^{(i)})$. The class $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ is the essential closure of δ .

The essential closure of class of rings is always essentially closed.

Proposition 2.1. [9] Let δ be any class of rings which is not essentially closed. The essential closure $\delta_k = \bigcup_{i=0}^{\infty} \delta^{(i)}$ of δ is essentially closed

Proposition 2.2. [2] Let $*$ be the class of all $*$ -rings. The class $*_k = \bigcup_{i=0}^{\infty} *^{(i)}$ is the essential closure of $*$ and $*_k$ is a special class of rings.

Definition 2.2. [7] The upper radical $\mathcal{U}(\mu)$ of a special class μ of rings is called a special radical.

In fact, we have both the prime radical β and the upper radical $\mathcal{U}(*_k)$ are special radical classes, where $\mathcal{U}(*_k)$ is the upper radical of the essential closure $*_k$ of the class of all $*$ -rings. Hence, it is important to scrutinize the properties of $\mathcal{U}(*_k)$ and the structure of $*$ -ring especially for restricted graded version.

3 Graded $*$ -rings

Since the question which asked whether $\beta = \mathcal{U}(*_k)$ is still open, we will be trying to bring this open problem to the graded version. We start from the following definition.

Let \mathbb{Z} be the set of all integers. A ring R is called a \mathbb{Z} -graded if there is a family of subgroups $\{R_n\}_{n \in \mathbb{Z}}$ of R which satisfies

1. $R = \bigoplus_n R_n$, and
2. $R_n R_m \subseteq R_{n+m}, \forall n, m \in \mathbb{Z}$.

In general, we have the following definition.

Definition 3.1. [1] Let G be an additive group. A ring R is called a G -graded ring if $R = \bigoplus_{g \in G} R_g$ where the set $\{R_g | g \in G\}$ is the additive subgroups of R such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

Example 3.1. For every ring A . The polynomial ring $A(x)$ over A is a graded ring by its degree.

Further implementation of graded rings in radical theory motivates the restricted graded radical class. In specific case, restricted graded Jacobson radical class can be explored in [13].

Definition 3.2. A ring A is called a graded $*$ -ring if $A = \bigoplus_{g \in G} A_g$, where the set $\{A_g | g \in G\}$ is the additive subgroups of A such that $A_g R_h \subseteq A_{gh}$ for all $g, h \in G$ and it is a $*$ -ring or simply A is a graded ring which is a $*$ -ring.

Remark 3.1. It is clear that every polynomial ring $R(x)$ over a ring R can be seen as a graded ring by its degree. Furthermore, if R is a semiprime ring, then so is $R(x)$. Now, a natural question asks whether the polynomial ring $R(x)$ over $*$ -ring R is also a $*$ -ring?

The answer of the question described in the Remark 3.1 will be explained in the following proposition.

Proposition 3.1. Let R be a $*$ -ring with unity. The polynomial ring $R(x)$ is not a $*$ -ring.

Proof. Let R be a ring with unity which is $*$ -ring. Then it follows from the definition of $*$ -ring that R is a prime ring and $\beta(R/I) = R/I$ for every nonzero ideal proper I of R . Now, suppose $R(x)$ is the polynomial ring over R . It follows from [12] that $R(x)/\langle x \rangle \cong R$ which implies that $\beta(R(x)/\langle x \rangle) = 0$ since, R is a prime ring. Thus, $R(x)$ is not a $*$ -ring. \square

So, we do not have any natural graded $*$ -rings constructed by polynomial rings.

In this part, we provide a graded $*$ -ring.

Theorem 3.1. Let \mathbb{R} be set of all real numbers. Then the ring $M_{2 \times 2}(\mathbb{R})$ of all 2×2 matrices over \mathbb{R} is a graded $*$ -ring.

Proof. Let \mathbb{R} be set of all real numbers and let δ be the class of all simple rings with unity. Then $\mathbb{R} \in \delta$. It follows from Example 3.8.14 in [7] that δ is a special class of rings which implies δ consists of prime rings. Thus, the ring \mathbb{R} is a prime ring. It is clear that \mathbb{R} is a $*$ -ring since \mathbb{R} is simple. Furthermore, it follows from Example 11.2.15 in [12] that the ring $M_{2 \times 2}(\mathbb{R})$ of all 2×2 matrices over \mathbb{R} is a simple ring with unity. Thus $M_{2 \times 2}(\mathbb{R}) \in \delta$. Hence, $M_{2 \times 2}(\mathbb{R})$ is a $*$ -ring. \square

In fact, the speciality of the class π of all prime rings implies that the essential closure π_k of π is the class π itself since π is closed under essential extension. A radical class γ is said to have the Amitsur property if $\gamma(A(x)) = (\gamma(A))(x)$ for every ring A [5]. Furthermore, it follows from the Amitsur property of β that if A is a prime ring, then the polynomial $A(x) \in S\beta$, where $S\beta$ is the semisimple class of the prime radical β . We continue to compare with property of the class $*$ of all $*$ -rings.

Proposition 3.2. *The class $*$ of all $*$ -rings is properly contained in the semisimple class $SU(*_k)$ of $\mathcal{U}(*_k)$.*

Proof. It is clear that $*$ is contained in $*_k$. Now let R be any $*$ -ring. This means that $R \in *$. It follows from Proposition 3.1 that the polynomial ring $R(x)$ is not a $*$ -ring. Thus, $R(x) \notin *$. We will show that $R(x) \in SU(*_k)$.

Since $R \in *$, $R \in *_k$. It follows from Corollary 13 in [5] that $R(x) \in SU(*_k)$. This implies $*$ is properly contained in $SU(*_k)$ which ends the proof. \square

Now, we shall follow the construction of the graded radical in G -category introduced by [1]. Let G be any arbitrary group and let A be a G -graded ring. The graded prime radical of A is denoted by $\beta_G(A)$ and it is defined as the intersection of all the G -graded prime ideals of A . Furthermore, the class of rings

$$\beta_G = \{A \mid A \text{ is a } G\text{-graded and } \beta_G(A) = A\}$$

is the graded prime radical class determined by the class of G -graded prime rings [1].

Definition 3.3. [1] *A nonempty class μ_G of G -graded rings is called a G -graded special class if satisfies the following conditions:*

1. γ_G consists of G -graded prime rings.
2. If $A \in \mu_G$, then every nonzero graded ideal I of A is in μ_G .
3. If A is any G -graded ring such that there exists a graded two-sided essential ideal I of A and $I \in \mu_G$ implies $A \in \mu$.

Let G be any group. Examples of G -graded special class of rings can be seen in the Example 3.2 and the Example 3.3 below.

Example 3.2. *Let G be any group. It is clear that the class π_G of all G -graded prime rings is a G -graded special class of rings.*

It is clear that the class π_G of all G -graded prime rings is the largest G -graded special class of rings. The Example 3.3 explains a specific example of G -graded special class of rings which is properly contained in the class π_G of all G -graded prime rings.

Example 3.3. Define $(*_k)_G = \{A \mid A \text{ is a } G\text{-graded ring such that } A^u \in *_k\}$, where $*_k$ is the essential closure of the class of all $*$ -rings which is previously described in the Proposition 2.2. Then class $(*_k)_G$ is graded special.

Remark 3.2. In specific case, when $G = \mathbb{Z}_4$, the class $(*_k)_{\mathbb{Z}_4}$ is not an empty set since $M_{2 \times 2}(\mathbb{R})$ is contained in $(*_k)_{\mathbb{Z}_4}$. In general, we can follow the construction of graded special radical [1] which is the graded upper radical of graded special class. Hence, for any group G . The graded radical $U((*_k)_G) = \{A \mid A \text{ is } G\text{-graded and no nonzero graded homomorphic image of } A \text{ in } (*_k)_G\}$.

We end this paper by arising the following question.

Question 1. For any group G , whether β_G coincide with $U((*_k)_G)$?, where β_G is the G -graded prime radical and $U((*_k)_G) = \{A \mid A \text{ is a } G\text{-graded ring and } A \text{ has no nonzero homomorphic image in } (*_k)_G \text{ (the class which is previously described in the Example 3.3 and Remark 3.2.)}$

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