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Numerical and Experimental for Optimal Sensor Location in Lumped and Distributed Parameter System

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***Abstract*—In estimating parameters, a small sample with high information content is preferable to a large sample but there is insufficient information. Therefore, the selecting of the optimal sensor location becomes a crucial problem in parameter estimation. The problem of determining an estimation through sampling data is a part of the inverse problems. Generally, the inverse problem has no solution in the ordinary sense because in most cases the known data have been contaminated with noises. The minimizer of the least square functional is usually taken as the solution to the inverse problem. This article demonstrates how to implement numerically the inverse problems in relation to parameter estimation when the noise set is generated independently during the experiment. The numerical simulation is applied to a distributed parameter system of the parabolic equation to find the optimal sensor locations for the parameter as well as to a model of a dynamic system to obtain the optimal time for measurements. Based on the result of the numerical experiment, it is found that different parameters in the same system could have different optimal samples.**

***Index Terms*—optimal sensor location, optimal design, FIM, parameter estimation.**

I. Introduction

TWo approaches commonly used to investigate the phenomenon of the real world phenomenon are through their mathematical representation [1] and by making replication of their behavior in laboratory scale [2]. On the mathematical representation, a relevant mathematical model is built and studied. In most cases, the mathematical model involves some parameters that require to be estimated through observation data. The information content of the parameters possessed by the sample is more important than the sample size itself. A small sample containing a lot of information about parameters is much more significant than another large sample but lacks information content. Assessing the informational content of observational data is a crucial issue in parameter estimation [3], [4]. The information content about existing parameters in a sample is usually measured through the criteria that minimize some cost functional over the set Fisher information matrices (FIM's) [5]. The FIM plays an important role in the optimal design of parameter estimation since the Cramer-Rao lower bound inequality can only be attained by the inverse of FIM [6], [7].

Initially, the problem of parameter estimation appeared in statistics where the functional relationship among variables is presumed to have an explicit representation like linear, quadratic, exponential, or their combination [8], [9]. The set of parameters in the statistical model was assumed to arise from a population with a certain distribution probability [6], [9]. However, in a distributed parameter system, the relationship between variables tends to be more complicated. Nowadays, the explicit models mentioned earlier are inadequate anymore to deal with problems in science and engineering [10], [11], [12]. The implicit relationship among variables represented in more complex operator equations such as differential equations subjected to initial or boundary value conditions, integral equations, or integro-differential equations are well suited and had attracted the attention of many mathematicians and applied scientists, see [4], [13], [14], [15].

The term distributed-parameter system (DPS) refers to dynamical systems whose state depends not only on time but also on spatial coordinates [4], meanwhile the lumped-parameter system (LPS) depends only on time [6]. The abstract formulation of parameter estimation (identification) in the parameter system can be described by the following operator equation [17]:

$$G(\theta) = d, \tag{1}$$

where G denotes an operator, θ constitutes the parameter vector characterizing the model and d represents the observation data that might be a function of time or a discrete set of observations. The problems in the parameter identification consist of the forward and the inverse problem [7], [14]. On the forward problem, the parameter θ is given, while the output model is determined. Conversely, the inverse problem is addressed to obtain an estimate for parameter θ through the observation data d [18]. In this formulation, the problem of solving an operator equation and estimating some parameters must be solved simultaneously. The inverse problem is generally ill-posed because of the noise disturbance most probably taken into account in the observation of data due to an incorrect reading of the instrument or a rounding error of the numerical data [19]. This is the main obstacle in solving the inverse problem in connection with parameter estimation. This situation can be represented as follows [17].

$$\begin{aligned} \text{data} &= \text{clean observation+noise component} \\ d &= G(\theta_{true}) + \epsilon \qquad (2) \\ &= d_{true} + \epsilon \qquad (3) \end{aligned}$$

where $d_{true} = G(\theta_{true})$, i.e. fulfills exactly equation (1) whenever $\theta = \theta_{true}$. This exact parameter is sometimes referred to as the nominal or natural parameter and it presumed

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to exist naturally. In most cases the nominal parameter is not available explicitly, instead, it is approximated by an estimate obtained through the observational data (sample) [17], [15].

An intuitive consideration in choosing the optimal sample is that a sample containing much information for some parameter is supposed to result in a better estimate for that parameter in the sense more accurate and more reliable. Accurate refers to the proximity of the nominal parameter, while reliability refers to the small variances. The premise used in parameter estimation is that the information contained on the parameter θ may vary considerably from one-time measurement to another [20]. The purpose of determining the optimal sensor location is intended to select a data set that contains the most information about parameter to be estimated [21], [22].

One of the parameter model that frequently appears on the applied sciences is the system of differential equations which models some physical, sociological or biological phenomenon [24].

$$\dot{x}(t) = g\left(t, x(t), \theta\right), t \geq t_0, x(t_0) = x_0(\theta), \quad (4)$$

where $x(t) \in \mathcal{R}^n$ denotes the vector of state variables and $\theta \in \mathcal{R}^p$. This model is a kind of LPS since the state is solely on the time variable. For a given the admissible parameter θ, the solution $x = x(t, \theta)$ to the initial value problem (4) can be obtained. In most cases, the model output emerges in the form of functional with respect the state variable $x(t, \theta)$, i.e. $f(t) := h(t, x(t, \theta), \theta)$. Consequently, the model output (4) can be represented as

$$z(t) = f(t, \theta), t \geq t_0. \quad (5)$$

On the other hand, the DPS model can be expressed by a partial differential equation subjected to some initial and boundary values [4]:

$$\frac{\partial y}{\partial t} = \mathcal{F}\left(x, t, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_2^2}, \theta\right), \quad (6)$$

where $x \in \Omega \subset \mathcal{R}^2, t \in (0, T)$ and boundary conditions $\mathcal{B}\left(x, t, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \theta\right) = 0, x \in \partial\Omega, t \in (0, T)$ and initial value $y(x, 0) = y_0(x), x \in \Omega$.

In the numerical implementation, this system is generally solved by a semi-discrete method in which one variable is discretized while maintaining another continuous. Thus, the output model given by (5) for LPS in certain circumstances can be adapted to the DPS. By assumption the model (4) is representative enough concerning the real model, the existence of the nominal parameter $\theta_0 \in \mathcal{A}$ is guaranteed, and thus the system output $y(t) = f(t, \theta_0)$ can be regarded as the true output. It is assumed that the measurement $z(t)$ at time t is contaminated by noise. This means that the measurement consists of two components, i.e. the system output (clean data) and the error component (noise).

$$z(t) = f\left(t, \theta_0\right) + \epsilon(t), t \in [0, T]. \quad (7)$$

The random processes of measurement error $\epsilon(t)$ is assumed to have zero mean, time-dependent variance $\sigma^2(t)$, independent within the measurements.

This paper is structured as follows: section 2 gives a brief overview of the optimal sensor location through some simple illustrations. Section 3 presents the abstract formulation of the problem in order to obtain the optimal sensor location which is referred to as the optimal experimental design or the optimal design method. Some famous optimal criteria are introduced in this section. In section 4, some numerical simulations of inverse problems are conducted for both DPS and LPS through a series of experiments in which the noises are generated independently during the experiment. The estimate $\hat{\theta}$ for θ_0 is taken as the mean of estimates obtained from all experiments as performed in the Monte Carlo simulation. The sample's quality is measured through its accuracy and variabilities.

II. Optimal Sensor Location: An Overview

During the COVID-19 pandemic, all public areas including schools and universities demand the visitors to check their body temperature using thermometer sensors placed at the entrances. Once the author took temperature measurements in three different locations through the palms, back of hands, and forehead. The results were 36.1, 36.2, and 36.4, respectively. Which value accurately represents my body temperature at that time? This is the fundamental question on the problem of optimal sensor location.

For the next illustration, consider the explicit model given by the following equation.

$$f(x, t; \theta) = \exp(-\theta\pi^2 t)\sin(2\pi x)\cos(20x). \quad (8)$$

In order to get a better understand these issues, consider the model (8) with $t = 0.2$ and nominal parameter $\theta_0 = 1$. Think of the model output that was mixed by noise contamination as shown in Figure 1. The noise set was generated by MATLAB using `randn` function.

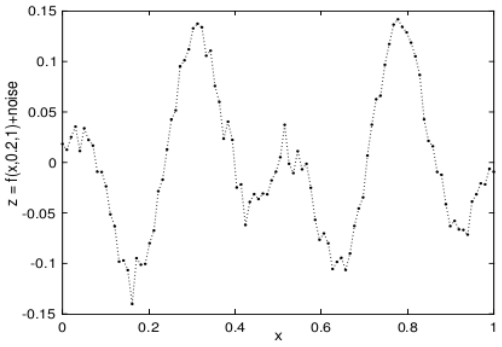

Fig. 1. Model output of explicit model (8) with noise contamination.

To estimate the nominal parameter θ_0, a sample should be taken through the sensor location which is then used to define the sum square errors (SSE). For example, if x_1 and x_2 are taken as sensor points then the following SSE must be minimized.

$$\mathcal{J}(\theta) = \left(f(x_1, t; \theta) - z_1\right)^2 + \left(f(x_2, t; \theta) - z_2\right)^2,$$

where z_1 and z_2 are measurements with noise contamination as given in (7). The minimizer of SSE is set as the estimate given by x_1 and x_2. Let Ω be the candidate sensor location which consists of 100 points evenly distributed on the interval $[0, 1]$. The experimental results were summarized in Table I.

TABLE I
ESTIMATES OBTAINED FROM SEVERAL SENSOR LOCATIONS.

Sensor	Points of measurement	Estimate $\hat{\theta}$
1	30 points uniformly	1.3580
2	10 first points	1.0009
3	10 last points	1.5000
4	$x_1 = 0.4545, x_2 = 0.5455, x_3 = 0.6162$	1.2610

Based on the experiment it is clear that the number of points within sensors does not guarantee a better estimate. For example, sensor 1 consists of 30 points which is worse than sensor 3 which contains only 10 points. According to [4], the number of required sensor points in some cases is just a little more than the number of parameters so that it can be kept to a minimum. To verify this statement, numerical experiments have also been carried out for each sensor point-wise. The results are shown in Figure 2. The most accurate is achieved by $x = 0.7172$ which produces $\hat{\theta} = 1.0818$. Surprisingly, a single point contains more information about parameter θ than several points collectively. In case the nominal parameter θ_0 was not assumed beforehand, the accuracy cannot be measured. Instead, the quality of the estimate is measured by the reliability of the sensor point which is indicated by the standard error of the estimates throughout the simulation.

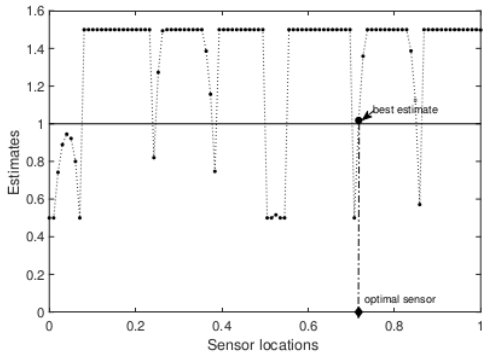

Fig. 2. Estimates obtained by single point sensors.

III. THE OPTIMAL DESIGN METHOD

Considering the output model $f(t,\theta)$ and the statistical model $z(t)$ (observation processes), the objective function is given by a generalized functional error as

$$J(y,\theta) := \int_0^T \frac{1}{\sigma^2(t)} \left(y(t) - f(t,\theta)\right)^2 dP(t) \qquad (9)$$

where P denotes a general measure defined on σ-algebra of $[0,T]$. In particular, P is the probability measure. Let $\tau = \{t_1, \cdots, t_N \ : \ t_1 < t_2 < \cdots < t_N\}$. The following Dirac measure δ_{t_i} is one simple but important probability measure, where $\delta_{t_i}(A) = 1$ if $t_i \in A$ and $\delta_{t_i}(A) = 0$ if $t_i \notin A$. For a given set of points $\tau = \{t_1, t_2, \cdots, t_N\}$, the measure P_τ is defined as

$$P_\tau = \sum_{i=1}^{N} \delta_{t_i}. \qquad (10)$$

Employing this measure, the cost functional (9) reduces to the discrete version of the weighted least squared functional.

$$J_d(y,\theta) = \sum_{i=1}^{N} \frac{1}{\sigma^2(t_i)} \left(z(t_i) - f(t_i,\theta)\right)^2. \qquad (11)$$

Although the continuous version (9) is applicable numerically, for instance by using some quadrature formula for integral approximation, the discrete version (11) is more beneficial for numerical implementation. Therefore, this experiment will utilize the discrete version.

According to [5], the abstract problem of the optimal design is choosing "the best" measure $\hat{P}$ over $\mathcal{P}[0,T]$ the set of all probability measures on $[0,T]$ in the sense

$$\mathcal{J}\left(F(\hat{P},\theta_0)\right) = \min_{P \in \mathcal{P}(0,T)} \mathcal{J}\left(F(P,\theta_0)\right), \qquad (12)$$

where $F(P,\theta_0)$ denotes the Fisher information matrix (FIM) given by

$$F(P,\theta_0) = \int_0^T \frac{1}{\sigma^2(s)} \nabla_\theta^T f(s,\theta_0) \nabla_\theta f(s,\theta_0) dP(s). \qquad (13)$$

Practically, the probability measure P is characterized by mesh, or points in τ as aforementioned such that the optimal design method can be transformed into the optimization problem of choosing a set of points $\tau^* = \{t_i^*\}$, $i = 1, \cdots, N$ such that the following condition holds.

$$\mathcal{J}\left(F(\tau^*,\theta_0)\right) = \min_{\tau \in \mathcal{T}} \mathcal{J}\left(F(\tau,\theta_0)\right), \qquad (14)$$

where $\mathcal{T}$ stands for a set of all meshes $0 \le t_1 \le \cdots \le t_N \le T$.

The discrete version of FIM had been introduced by Thomaseth and Cobelli [25] as

$$F(\tau,\theta_0) = \sum_{i=1}^{N} \frac{1}{\sigma^2(t_i)} \nabla_\theta^T f(t_i,\theta_0) \ \nabla_\theta f(t_i,\theta_0), \qquad (15)$$

where $\nabla_\theta f := \frac{\partial f(t,\theta)}{\partial \theta}$ denotes the vector gradient of parameters. Throughout this article, the simulations use the discrete version of FIM's (15) as the arguments of the cost functional (14). Furthermore, the current objective is to find an s^* which minimizes some real-valued function of $\mathcal{J}$ defined on all possible FIMs. Some optimal criteria were considered in [4], [5], [20], among others are:

- D-optimal, which maximizes the determinant of FIM, i.e.
$$\mathcal{J}(F) := -\ln\det(F)$$
where "det" denotes the determinant of matrix.
- E-optimal, which maximizes the spectral radius of FIM, i.e.
$$\mathcal{J}(F) := \lambda_{\max}(F^{-1})$$
where $\lambda_{\max}$ is the largest eigen value.
- A-optimal or SE-optimal, which maximizes the trace of FIM, i.e.
$$\mathcal{J}(F) := \mathrm{tr}(F^{-1})$$
where "tr" stands for the trace of matrix, viz. the sum of entries on the main diagonal.

In the first stage, the optimal sample is determined according to functional (14) based on some optimal criteria. Secondly, the estimation of parameter θ is taken as the minimizer of the

least squared functional (11). There is no doubt that the larger the sample size and the more parameters involved, the more difficult it is to solve the problem numerically because of the high-dimension problem. On the other hand, in classical parameter estimation, a small sample size may contain less information about the parameters to be estimated. However, in the optimal design, the sample size can be reduced as long as the content of the information inherited by sample can be kept at maximum [4].

In the LPS, the optimal sample corresponds to minimizing the number of measurements and choosing the best time when the measurements should be carried out. It is similar to determination of the best position for measurement in the case of spatial variables in the DPS. In the dynamical system or LPS, the traditional sensitivity function (TSF) and the generalized sensitivity function (GSF) are often used as a reference in determining the optimal times of measurement [26], [20]. In the DPS, the optimal measurement relates to selecting a spatial point that meets the optimal criteria within a predetermined time horizon.

IV. Numerical Simulation

The first simulation takes a simple case in DPS, the equation of temperature distribution of a rod thin [4].

Example 1

The temperature distribution $y = y(x,t)$ on a rod thin with assumption no heat source and both ends with perfect insulation

$$\frac{\partial y}{\partial t}(x,t) = \theta \frac{\partial^2 y}{\partial x^2}(x,t), \quad x \in (0,1),\ t \in (0,t_f) \tag{16}$$

$$y(x,0) = \sin(\pi x) \tag{17}$$

where the parameter θ stands for the diffusivity of the material forming the rod. Suppose the two ends associate with $x = 0$ and $x = 1$, respectively, and at time $t = 0$ suddenly placed in contact with ice packs at 0° and that the temperature at the ends is maintained at all later times. The situation can be represented by the boundary conditions

$$y(0,t) = y(1,t) = 0,\ t \in (0,t_f). \tag{18}$$

It is deliberately the model was taken trivially so that it has an explicit solution given by

$$y(x,t) = e^{-\theta \pi^2 t} \sin(\pi x). \tag{19}$$

The objective here is to find x^1 the location which contains the most information about parameter θ. Theoretically, it is required to solve the optimization problem with the objective function $\mathcal{J}$ defined on the set of FIM's. However, at this moment the potential sensor locations are prior defined intuitively to be candidates for the optimal sample. To allow numerical simulation does work, the spatial variable is discretized and the time variable is kept continuous so that the observation process can be represented by

$$z(t) = y_m(t) + \varepsilon_m(t),\ t \in T := (0,t_f)$$

where

$$\begin{aligned} y_m(t) &= \text{col}\left(y(x^1,t), y(x^2,t), \cdots, y(x^n,t)\right) \\ \varepsilon_m(t) &= \text{col}\left(\varepsilon(x^1,t), \varepsilon(x^2,t), \cdots, \varepsilon(x^n,t)\right). \end{aligned}$$

In this case, $z(t)$ is the observation vector in a n-dimenstion space, $x^j \in [0,1], j = 1, \cdots, n$ are the sensor location $y(x^j,t)$ is regarded as the true output at position x^j and time t, and $\varepsilon = \varepsilon(x,t)$ is the noisy random process with assumption $E(\varepsilon_m(t)) = 0$, $\text{Var}(\varepsilon_m(t)) = \sigma^2(t)$, $\text{Cov}(\varepsilon_m(t)\varepsilon_m(s)) = \sigma(t)\sigma(s)\delta(t-s)$ for $t \in T$, where δ is the Dirac distribution concentrated at origin.

Before solving the inverse problem by a certain optimization algorithm, the observation data were disturbed by some artificial noise. For each point of optimal sensor x^j, the experiments are carried out repeatedly with various noises. The noises are generated by MATLAB toolbox randn. The following algorithm is designed to write computer codes on MATLAB.

Algorithm 1

1) Discretize the time interval $[0,t_f]$, e.g. by a uniformly partition with step size Δ_t. Let $T = \{t_1, t_2, \cdots, t_{n_t}\}$ be the measurement times where n_t is the number of sampling times.
2) Define the set of potential sensor location $\Omega_0 = \{x_1, x_2, \cdots, x_N\}$, for instance by taking the equidistance points (uniformly).
3) For each $i = 1, 2, \cdots, N$:
   a) Represent the vector of true or system output, i.e. $y(t, x_i; \theta)$ for $\theta = \theta_0$. So, for each $i = 1, 2, \cdots, N$, the true output is represented by
   $$y_i = (y_i^k | k = 1, \cdots, n_t)$$
   where $y_i^k = y(x_i, t_k)$.
   b) Generate the noise $\epsilon_i \sim N(0, \sigma^2)$
   $$\epsilon_i = (\epsilon_i^k | k = 1, \cdots, n_t).$$
   c) Define the observation data $z_i = (z_i^k | k = 1, \cdots, n_t)$ by taking
   $$z_i(t_k) = y(x_i, t_k) + \epsilon_i^k.$$
   d) Define the functional error as the objective function $\mathcal{J}_i : \Theta_{ad} \to \mathcal{R}$ with
   $$\mathcal{J}_i(\theta) := \sum_{k=1}^{n_t} |z_i(t_k) - y(x_i, t_k; \theta)|^2.$$
   e) Corresponding to $x_i \in \Omega$, the estimate $\hat{\theta}_i$ is determined according to
   $$\hat{\theta}_i = \text{argmin}\left\{\mathcal{J}(\theta) | \theta \in \Theta_{ad}\right\}.$$
   f) Do experiment (a)-(e) as L times. The noises are distinguished from one experiment to another.
   g) Apply the principle on Monte Carlo simulation, the estimate corresponding to $x_i \in \Omega$ is taken as the mean of L estimates obtained through all experiments.
   h) Calculate the standard error of the estimate to measure its reliability.

The sensor location x^j corresponds to the smallest standard error is regarded as the most informative sample.

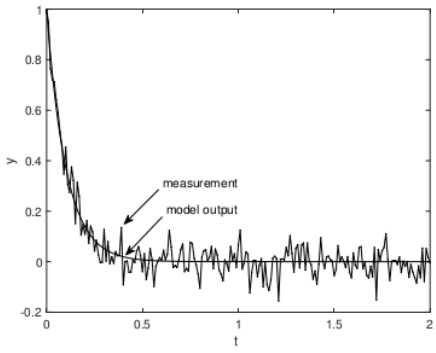

Fig. 3. The model outputs of Example 1 for $x = 0.5$ and their observation.

Numerical Experiment 1

For this simulation, it is assumed that $\theta_0 = 1$, $t_f = 2$, $\epsilon \sim N(0, \sigma^2)$ where $\sigma = 0.04$ is the random noises generated by MATLAB using randn. For example, Figure 3 shows the exact or true model output (smooth curve) and the observation or measurement data (irregular curve) for $x = 0.5$. The sampling had taken uniformly with $\Delta t = 0.01$ including endpoints. So, there will be as many as 201 measurements for each sensor location. Supposedly, the sensor locations being observed are given in the set

$$\Omega_0 = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}.$$

The experiment was performed as $L = 20$ repetitions where the noises were generated independently. The optimization problem can be solved by MATLAB, viz. function fminsearch and fminbnd of MATLAB optimization toolbox. The present numerical experiment adopted the fminbnd for dealing 1-D optimization problem. The standard error is calculated by the following formula.

$$SE_\theta(x) = \sum_{k=1}^{L} (\hat{\theta}_k(x) - \theta_0)^2.$$

The experimental results are presented in the three following consecutive figures. Figure 4 indicates the errors of each

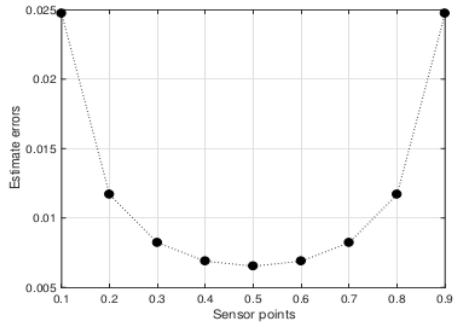

Fig. 4. Error of estimates for each sensor location.

estimate obtained from each sensor point. It is found that the most accurate estimator is given by sensor $x = 0.5$. The other sensor points generate estimates with symmetrical-like errors with respect $x = 0.5$. For instance, the accuracies given by the pairs $x = 0.1$ and $x = 0.9$, $x = 0.2$ and $x = 0.8$, $x = 0.3$ and $x = 0.7$, and $x = 0.4$ and $x = 0.6$ are look like similar.

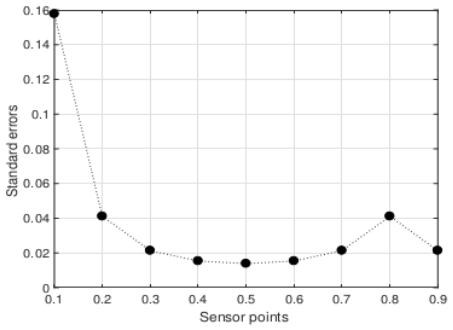

Fig. 5. Standard error of estimates for each sensor location.

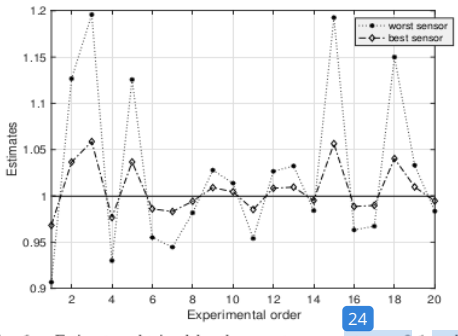

Fig. 6. Estimates obtained by the worst sensor at $x = 0.1$ and the best sensor at $x = 0.5$.

The standard errors for each estimate are shown in Figure 5. The smallest variance also happened at $x = 0.5$. This means that the sensor at $x = 0.5$ provides the most information about the parameter θ. In addition, although both $x = 0.1$ and $x = 0.9$ have comparable accuracy, the sensor at $x = 0.9$ is considered better because its standard error is smaller. Overall, it is concluded that the best sensor is given by $x = 0.5$ while the worst is provided by $x = 0.1$. The performance of two particular sensors in producing the estimates is displayed in Figure 6. Observe that the fluctuations provided by the worst sensor are significantly higher than the best sensor. In addition, two sets of estimates exhibit a strong linear correlation with a coefficient 0.9987.

Furthermore, it should be confirmed in the theoretical background that the optimal criteria must met by $x = 0.5$. By means of elementary calculus, the following FIM can be obtained easily.

$$M(x) = \underbrace{\frac{\pi^4}{\sigma^2} \int_0^{t_f} t^2 e^{-2\theta\pi^2 t}\, dt}_{>0} \sin^2(\pi x).$$

Because a single point is only considered, the FIM becomes a matrix of single element so that the optimal criteria such as E-optimal or A-optimal are trivial. Indeed, the determinant,

the eigenvalue, and the trace of 1×1 matrices is nothing but that element itself. Thus, the maximizer of optimal criteria on FIM is given x^j itself. In this case, $M(x^j) = k \sin^2(\pi x^j)$ where $k = \frac{\pi^4}{\sigma^2} \int_0^{t_f} t^2 e^{-2\theta\pi^2 t} dt > 0$. The matrix $M(x^j)$ reaches maximum at $\pi x^j = \pi/2$ or $x^j = 0.5$. Consequently, the D-optimal criteria $\mathcal{J}(M(x)) = -\ln \det M(x)$ reaches the minimum at $x = 0.5$. A similar argument for other optimal criteria. Hence, the result of numerical simulation fits perfectly with the theoretical background.

The following simulation takes the model of dynamical system of logistic population growth [5].

Example 2

Model of Verhulst-Pearl logistic describes the dynamical growth of one population which has the intrinsic growth and growth with saturation due to the carrying capacity.

$$\dot{x}(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right), \; x(0) = x_0 \tag{20}$$

where K denotes the carrying capacity, r stands for the intrinsic growth, and x_0 indicates the initial size of the population.

The corresponding parameters system given by $\theta := (K, r, x_0)$. In brief, the simulation considers only two parameters r and x_0 while K is considered as a constant. Let's take $K = 17.5$ so that the analytic solution of the model can be written as

$$x(t) = \frac{17.5}{1 + (\frac{17.5}{x_0} - 1)\exp(-rt)}. \tag{21}$$

The main issue here is how to choose the measurement points (times) so that the collected data contains as much as possible information about parameters x_0 and r. Practically, the problem should be breakdown into how to determine the duration of measurements, the number of measurements, and the distribution of measurements. The measurement data are presumed to have been contaminated by a set of noises. The inverse problem is solved repeatedly for each set of measurement data (observation process). The following algorithm is prepared for writing the MATLAB codes for numerical simulation.

Algorithm 2

1) Define the collection of the set of measurement times, $\Omega_0 := \{\tau_1, \tau_2, \cdots, \tau_N\}$ where $\tau_i = \{t_{i1}, t_{i2}, \cdots, t_{in}\}$. This means that for each τ_i, $i = 1, 2, \cdots, N$, a sample of size n is obtained via measurement at $t = t_{ij}, j = 1, 2, \cdots, n$.
2) Fix L, the number of repetitions.
3) Generate M, the randomly matrix with size $L \times n$ where the rows represent the vectors of noises of zeros mean, i.e. $\varepsilon \sim N(0, \sigma^2)$.
4) For each $\tau_i \in \Omega_0$: $i = 1, 2, \cdots, N$:
   a) Define the true output $x = (x(t; \theta_0) | t \in \tau_i)$, where θ_0 is the assumed nominal parameter.
   b) For each $k = 1, \cdots, L$, define the observation data $z_i^k(t)$ composed of the true output and the noise component, i.e. $z_i^k(t) = x(t; \theta_0) + \epsilon^k$ where $t \in \tau_i$ and ε^k emerged from the $k^{th}-$row of matrix M.
   i) Define the functional $\mathcal{J}_i^k : \Theta_{ad} \to \mathcal{R}$ with
   $$\mathcal{J}_i^k(\theta) := \sum_{t \in \tau_i} |z_i^k(t) - x(t; \theta)|^2.$$
   Apply some optimization technique to obtain $\hat{\theta}_i^k = \text{argmin}\left\{\mathcal{J}_i^k(\theta) | \theta \in \Theta_{ad}\right\}$. Furthermore, the notation $\hat{\theta}_k(\tau_i) =: \hat{\theta}_i^k$ denotes the estimate obtained through sample τ_i on the $k^{th}-$ repetition.
   ii) Finally, the estimate corresponding for the time measurement $\tau_i = \{t_{i1}, t_{i2}, \cdots, t_{in}\}$ is taken as the mean of estimates obtained from all repetitions, i.e.
   $$\hat{\theta}_i := \frac{1}{L} \sum_{k=1}^{L} \hat{\theta}_k(\tau_i).$$
   iii) Calculate the standard error of this estimate to know its variability by formula
   $$SE_\theta(\tau_i) = \sum_{k=1}^{L} (\hat{\theta}_k(\tau_i) - \theta_0)^2.$$

Hence, there will be N estimates generated from N measurement samples. The best sample is the sample corresponding to the estimate that has the smallest standard error dan the most accurate (in case the nominal parameter is known).

Numerical experiment 2

The experiment strategy fullfilled the constrained optimization as introduced in Banks et. al. [5] with a variation on the scenario to find the most informative sample. In this case, a total of 13 subintervals were prepared for measurements are shown in Table II. The 5 points which were distributed evenly are taken from each subinterval. The measurement data obtained through these points are used to estimate θ. For that purpose, the nominal parameter had been chosen, i.e. $r = 0.7$, $x_0 = 0.1$, and $L = 50$. The noises matrix M was generated by MATLAB with $\mu = 0$ and $\sigma = 0.125$. Similar to Experiment 1 before, the disturbance was a kind of Gaussian noise. The parameters estimation based on the non-Gaussian noises, i.e. the Lêvy noises had been implemented in the parameter estimation for models in the finance mathematics [27], [28].

TABLE II
SOME INTERVALS FOR MEASUREMENT

No	Interval	No	Interval	No	Interval	No	Interval
1	[0, 4]	5	[16, 20]	9	[0, 10]	13	[12, 20]
2	[4, 8]	6	[2, 8]	10	[10, 20]		
3	[8, 12]	7	[8, 16]	11	[0, 25]		
4	[12, 16]	8	[16, 22]	12	[5, 10]		

The MATLAB optimization toolbox fminsearch had been applied with the same initial values for all experiments. The numerical experiment results are summarized in Table III. The results in this table may be slightly different if other disturbance data are attached but their patterns might be similar. According to Table III, it is clear that the worst estimate was produced by sample 5 and also sample 8. They are not only lack in accuracy but also bad in standard errors.

TABLE III
THE ESTIMATES AND THEIR STANDARD ERRORS FOR EACH SAMPLE

Sampel Id	Estimates		Standard errors	
	r	x_0	r	x_0
1	0.6954	0.1048	0.4971	0.0495
2	0.6995	0.1005	0.0119	0.0047
3	0.6996	0.1022	0.0278	0.0238
4	0.7075	1.0379	2.7343	390.79
5	1.3641	3.7687	83.300	2696.8
6	0.6985	0.1012	0.1220	0.0049
7	0.7001	0.1018	0.0290	0.0235
8	1.4344	3.6315	84.200	2474.0
9	0.6993	0.1010	0.0112	0.0054
10	0.7068	0.1344	0.3802	0.8510
11	0.6988	0.1039	0.0832	0.0320
12	0.7002	0.1000	0.0084	0.0043
13	0.7378	1.2389	18.5817	18.5910

TABLE IV
THE VALUES OF OPTIMAL CRITERIA FOR EACH SAMPLE

Sample Id	D-optimal	E-optimal	SE-optimal
1	-5.8509	0.8775	0.8808
2	-12.8318	0.0220	0.0221
3	-11.4845	0.0618	0.0619
4	-2.5850	10.4363	10.4435
5	8.4335	3858.4	3859.6
6	-12.3877	0.0248	0.0250
7	-10.6975	0.0865	0.0868
8	8.8811	5131.5	5132.9
9	-9.6012	0.4546	0.4548
10	-6.4705	1.4182	1.4193
11	-9.4171	0.1657	0.1662
12	-13.2531	0.0152	0.0153
13	-1.7538	18.5817	18.5910

To confirm the experimental results to the theoretical background, the Fisher information matrix (FIM) needs to be calculated for each sample by the formula (15) with $\sigma(t_i) = 1$ for all $i = 1, 2, \cdots, N$, then the corresponding optimal criteria (D, E, and SE-optimal) are evaluated. The results are displayed in Table IV. Based on the evaluation, the optimal criteria values of samples 5 and 8 are much bigger than the others. This fact corresponds to the poor accuracy and standard errors of estimates shown in Table III.

Theoretically, the smaller value of optimal criteria, the closer variance to the lower bound of Cramer-Rao inequality, and consequently the better of estimate. This condition is fullfiled by sample 2 and sample 12. Both samples reached not only the minimum value for all optimal criteria (Table IV) but also attained the highest accuracies and the smallest standard errors for both parameters r and x_0 (Table III). This suggests that both samples contain much more information about parameters than others.

In addition, it was found that a sample could contain much information about certain parameters but less for others. According to Table III, for instance, sample 4 yields good estimate for r but it tends worse for x_0. Similar behavior also occurs on their standard errors.

V. CONCLUDING REMARKS

The numerical simulation of the inverse problem was applied to the problem of finding the optimal location for the sensor and the optimal time for measurement. According to simulations, the optimal criteria does play an important role to obtain a sample that contains much information about the parameters being estimated. Although the lower bound of Cramer-Rao inequality is almost impossible to be achieved, it is suggested to have a sample with a small optimal criterion value. A sample of this kind will produce good parameter estimates in the sense of accuracy and resistance due to a small disturbance of the measurement data.

The simulation used a single sample to estimate all parameters. As a result, the qualities of estimates are disparate among the parameters. This indicates that the parameters have their optimal sample that can differ from each other. The problem of determining a single sample that contains a lot of information for all parameters is a crucial issue in the optimal design of the measurement method.

There are still many problems that need to be addressed through further research. In case such an explicit form is not available then the numerical treatment will become more difficult. A large number of sensors will definitely affect the computational complexity and bring forth multimodal optimization problems. In case the parameters depend on time as well as spatial variables, it will be more highly complicated to determine the optimal sensor for parameter estimation.

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