Journal of Algebra and Its Applications
Vol. 22, No. 5 (2023) 2350229 (8 pages)

DOI: 10.1142/S0219498823502298

Rings of Morita contexts which are Dubrovin valuation rings

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Received 14 May 2022
Accepted 15 June 2022
Published 21 July 2022

Communicated by S. R. López-Permouth

Let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a ring of the Morita context. We define a concept of Dubrovin valuation modules for bi-modules V and W and give necessary and sufficient conditions for T to be a Dubrovin valuation ring in terms of R, S, V and W.

Keywords: Morita context; Dubrovin valuation module; Dubrovin valuation ring.

Mathematics Subject Classification 2020: 16K99, 16D20, 16N99

1. Introduction

Let R be a prime Goldie ring with quotient ring $Q = Q(R)$. R is called a Dubrovin valuation ring with M if

(i) There is an ideal M of R such that R/M is a simple Artinian ring.
(ii) For any $q \in Q \backslash R$ there are $r, r_1 \in R$ such that $rq, qr_1 \in R \backslash M$.

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The original definition given by Dubrovin [1] is different from the one above. However these two definitions are equivalent (see [2, Lemma 5.1]). It is well known that if R is a Dubrovin valuation ring, then $M = J(R)$, the Jacobson radical of R.

Let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a ring of Morita context which is a prime Goldie ring with quotient ring $Q(T)$, where R and S are prime Goldie rings with quotient rings $Q(R)$, $Q(S)$, respectively, V is an (R,S)-bimodule and W is an (S,R)-bimodule.

We define a concept of Dubrovin valuation modules for bi-modules V and W (see Sec. 2) and obtain the following theorem.

Theorem 1.1. *The following conditions on* $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *are equivalent*:

(1) T *is a Dubrovin valuation ring.*
(2) (i) R *and* S *are Dubrovin valuation rings and*
 (ii) $R = VW$ *and* $S = WV$.
(3) (i) V *is an* (R,S)*-Dubrovin valuation bimodule in* $Q(V)$ *with* V_1 *and* W *is an* (S,R)*-Dubrovin valuation bimodule in* $Q(W)$ *with* W_1 *such that* $WV_1 = W_1V$ *and*
 (ii) $R = VW$ *and* $S = WV$.

We refer the readers to [3, 4] for some elementary properties of Dubrovin valuation rings and for arithmetic ideal theory in rings of Morita contexts, respectively.

2. Dubrovin Valuation Rings

The aim of this section is to prove the main result being mentioned in the introduction. We denote by $J(R)$ the Jacobson radical of a ring R. It is known that $J(T)$ is of the following form:

$$J(T) = \begin{pmatrix} J(R) & V_0 \\ W_0 & J(S) \end{pmatrix},$$

where $V_0 = \{v \in V \mid vW \subseteq J(R)\}$ and $W_0 = \{w \in W \mid wV \subseteq J(S)\}$, which can be proven by using N-radical [5, 6]. It is clear that V_0 is an (R,S)-submodule of V and W_0 is an (S,R)-submodule of W. We begin with some properties for Jacobson radicals as shown in Lemmas 2.1–2.3.

Lemma 2.1 ([6]). *Let* $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *be a ring of Morita context. Then,*

(1) $VJ(S)W \subseteq V_0W \subseteq J(R), VJ(S) \subseteq V_0$ *and* $J(R)V \subseteq V_0$.
(2) $WJ(R)V \subseteq W_0V \subseteq J(S), WJ(R) \subseteq W_0$ *and* $J(S)W \subseteq W_0$.

Proof. (1) Since $J(T)$ is an ideal, we have

$$\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \begin{pmatrix} J(R) & V_0 \\ W_0 & J(S) \end{pmatrix} \subseteq \begin{pmatrix} J(R) & V_0 \\ W_0 & J(S) \end{pmatrix}.$$

Thus we have $VJ(S) \subseteq V_0$ and $VJ(S)W \subseteq V_0W \subseteq J(R)$. Since $J(R)VW \subseteq J(R)$, it follows that $J(R)V \subseteq V_0$.

(2) The proof is similar to one of (1). □

Lemma 2.2 ([6]). *Let* $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *be a ring of Morita context and suppose* $R = VW$ *and* $S = WV$*. Then,*

(1) $J(R) = VJ(S)W$ *and* $J(S) = WJ(R)V$.
(2) $J(R)V = V_0 = VJ(S)$ *and* $J(S)W = W_0 = WJ(R)$.

Proof. (1) $J(R) = VWJ(R)VW \subseteq VJ(S)W \subseteq J(R)$ by Lemma 2.1 and so $J(R) = VJ(S)W$ follows. Similarly, $J(S) = WJ(R)V$.

(2) $V_0 = V_0WV \subseteq J(R)V \subseteq V_0$ by Lemma 2.1 and so $V_0 = J(R)V$. Further $VJ(S) = VJ(S)WV = J(R)V = V_0$. Similarly, we have $J(S)W = W_0 = WJ(R)$. □

In the remainder of this section, we assume that T is a prime Goldie ring with its quotient ring $Q(T)$, a simple Artinian ring. Then we have the following properties:

(1) R and S are both prime Goldie rings with quotient rings $Q(R)$ and $Q(S)$, respectively.
(2) $Q(R)V = VQ(S)$, denoted by $Q(V)$ and $Q(S)W = WQ(R)$, denoted by $Q(W)$.
(3) $Q(T) = \begin{pmatrix} Q(R) & Q(V) \\ Q(W) & Q(S) \end{pmatrix}$ (see [4]).

We put: $\bar{T} = T/J(T)$, $\bar{R} = R/J(R)$, $\bar{S} = S/J(S)$,$\bar{V} = V/V_0$ and $\bar{W} = W/W_0$ unless otherwise stated. Then $\bar{V}$ is an $(\bar{R}, \bar{S})$-bimodule since V_0 contains $VJ(S)$ and $J(R)V$ by Lemma 2.1. Similarly, $\bar{W}$ is an $(\bar{S}, \bar{R})$-bimodule and so $\bar{T} = \begin{pmatrix} \bar{R} & \bar{V} \\ \bar{W} & \bar{S} \end{pmatrix}$ is a ring of Morita context.

Lemma 2.3. $\bar{T} = T/J(T)$ *is a simple Artinian ring if and only if*

(1) $\bar{R} = R/J(R)$ *and* $\bar{S} = S/J(S)$ *are simple Artinian rings.*
(2) $\bar{V}$ *is* $(\bar{R}, \bar{S})$*-simple, that is, there are no proper* $(\bar{R}, \bar{S})$*-submodules of* $\bar{V}$,${}_{\bar{R}}\bar{V}$ *is left Artinian and* $\bar{V}_{\bar{S}}$ *is right Artinian.*
(3) $\bar{W}$ *is* $(\bar{S}, \bar{R})$*-simple,* ${}_{\bar{S}}\bar{W}$ *is left Artinian and* $\bar{W}_{\bar{R}}$ *is right Artinian.*
(4) $J(R)$ *does not contain* VW *and* $J(S)$ *does not contain* WV.

Proof. It follows from the similar way to [7, Proposition 1.1.7] that $\bar{T}$ is right and left Artinian if and only if $\bar{R}_{\bar{R}}$, $\bar{S}_{\bar{S}}$, $\bar{V}_{\bar{S}}, \bar{W}_{\bar{R}}$ are right Artinian and ${}_{\bar{R}}\bar{R}$, ${}_{\bar{S}}\bar{S}$, ${}_{\bar{R}}\bar{V}$, ${}_{\bar{S}}\bar{W}$ are left Artinian.

Suppose that $\bar{T}$ is simple. If $\bar{R}$ is not simple, then there is an ideal $\bar{I}$ of $\bar{R}$ with $\bar{R} \supset \bar{I} \supset (\bar{0})$ and $\begin{pmatrix} \bar{I} & \bar{I}\bar{V} \\ \bar{W}\bar{I} & \bar{W}\bar{I}\bar{V} \end{pmatrix}$ is an nonzero ideal of $\bar{T}$, a contradiction. Hence $\bar{R}$ is simple and similarly $\bar{S}$ is simple. If $\bar{V}$ is not $(\bar{R}, \bar{S})$-simple, then there is an (R, S)-submodule V_1 of V with $V \supset V_1 \supset V_0$ and $A = \begin{pmatrix} V_1W & V_1 \\ WV_1W & WV_1 \end{pmatrix}$ is an ideal of

T such that $T \supset A$ and $J(T)$ does not contain A. Thus $T \supset A + J(T) \supset J(T)$, a contradiction. Hence $\bar{V}$ is $(\bar{R}, \bar{S})$-simple and similarly $\bar{W}$ is $(\bar{S}, \bar{R})$-simple.

If $J(R) \supseteq VW$, then $V = V_0$, that is, $\bar{V} = (\bar{0})$ and $\bar{T}$ is not simple. Hence $J(R)$ does not contain VW and similarly $J(S)$ does not contain WV.

Conversely suppose that (1)–(4) are satisfied. Then $\bar{T}$ is left and right Artinian. We prove that $\bar{T}$ is prime by using [4, Lemma 1.1]. If $\bar{v}\bar{W} = (\bar{0})$, where $v \in V$, then $vW \subseteq J(R)$ and so $v \in V_0$, that is, $\bar{v} = \bar{0}$. If $\bar{V}\bar{w} = (\bar{0})$, where $w \in W$, then $Vw \subseteq J(R)$ and $WVwV \subseteq WJ(R)V \subseteq J(S)$ by Lemma 2.1. Since $J(S) \not\supseteq WV$, $wV \subseteq J(S)$ and so $w \in W_0$, that is, $\bar{w} = \bar{0}$.

If $\bar{V}\bar{s}\bar{W} = (\bar{0})$, where $s \in S$, that is, $VsW \subseteq J(R)$ and so $Vs \subseteq V_0$. Suppose $s \notin J(S)$, then $SsS + J(S) = S$ and $VW = V(SsS + J(S))W \subseteq J(R)$ by Lemma 2.1, which contradicts the condition (4). Hence $\bar{s} = \bar{0}$. Hence $\bar{T}$ is a prime Goldie ring by [4, Lemma 1.1].

To prove that $\bar{T}$ is a simple ring, let $\bar{A} = \begin{pmatrix} \bar{I} & \bar{V_1} \\ \bar{W_1} & \bar{J} \end{pmatrix}$ be a nonzero ideal of $\bar{T}$, where I is an ideal of R, J is an ideal of S, V_1 is an (R, S)-submodule of V and W_1 is an (S, R)-submodule of W (see [4, Lemma 1.2 and Proposition 1.4]).

If $\bar{I} = (\bar{0})$, that is, $\bar{A} = \begin{pmatrix} (\bar{0}) & \bar{V_1} \\ \bar{W_1} & \bar{J} \end{pmatrix}$, then $\bar{A}\bar{T} \subseteq \bar{A}$ implies $\bar{V_1}\bar{W} = (\bar{0})$, that is, $V_1W \subseteq J(R)$. Thus $V_1 \subseteq V_0$ and hence $\bar{V_1} = (\bar{0})$.

If $\bar{J} \neq (\bar{0})$, then $J + J(S) = S$ and $\begin{pmatrix} \bar{0} & \bar{V} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} (\bar{0}) & (\bar{0}) \\ \bar{W_1} & \bar{J} \end{pmatrix} \subseteq \begin{pmatrix} (\bar{0}) & (\bar{0}) \\ \bar{W_1} & \bar{J} \end{pmatrix}$ implies $VJ \subseteq V_0$. Thus $V = V(J + J(S)) \subseteq V_0$ by Lemma 2.1, a contradiction. Hence $\bar{J} = (\bar{0})$ and then $\bar{W_1} = (\bar{0})$ since $\bar{T}$ is prime.

So we may assume that $\bar{I} \neq (\bar{0})$, that is, $\bar{I} = \bar{R}$. Symmetric argument shows $\bar{J} = \bar{S}$.

If $\bar{V_1} = (\bar{0})$, then $\bar{T}\bar{A} \subseteq \bar{A}$ implies $\bar{V} = (\bar{0})$, a contradiction. Hence $\bar{V_1} = \bar{V}$ and similarly $\bar{W_1} = \bar{W}$ follows. Hence $\bar{A} = \bar{T}$ and therefore $\bar{T}$ is a simple ring. $\square$

Remark 2.1. If R is a Dubrovin valuation ring, then $M = J(R)$, the Jacobson radical of R and the set of all R-ideals is linearly ordered by inclusion (see [3, Lemma 1.4.2 and Proposition 1.5.3]).

Lemma 2.4. $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *be a ring of Morita context and suppose that* T *is a Dubrovin valuation ring. Then*

(1) $R = VW$ *and* $S = WV$.
(2) R *and* S *are both Dubrovin valuation rings.*

Proof. First note that $\bar{R}, \bar{S}, \bar{V}$ and $\bar{W}$ are nonzero since $\bar{T}$ is a simple Artinian ring.

(1) If $WV \subseteq J(S)$, then $W \subseteq W_0$, a contradiction and so WV is not contained in $J(S)$. Similarly, VW is not contained in $J(R)$. Thus $\begin{pmatrix} VW & V \\ W & WV \end{pmatrix}$ is an ideal of T which is not contained in $J(T)$ and so $T = \begin{pmatrix} VW & V \\ W & WV \end{pmatrix}$, because the set of all ideals of T is linearly ordered by inclusion. Hence $VW = R$ and $WV = S$.

(2) By Lemma 2.3, $\bar{R}$ is a simple Artinian ring. To prove that R is a Dubrovin valuation ring, let $q \in Q(R)\backslash R$. There is a $t = \begin{pmatrix} r & v \\ w & s \end{pmatrix} \in T$ such that $T\backslash J(T) \ni \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & v \\ w & s \end{pmatrix} = \begin{pmatrix} qr & qv \\ 0 & 0 \end{pmatrix}$. If $qr \in R\backslash J(R)$, then we are done. If $qr \in J(R)$, then $qv \in V\backslash V_0$ and so $qvW \nsubseteq J(R)$ by Lemma 2.1. Hence $qvw \in R\backslash J(R)$ for any $w \in W$ with $qvw \notin J(R)$. Similarly, there is an $r \in R$ with $rq \in R\backslash J(R)$. Hence R is a Dubrovin valuation ring and a similar argument shows S is a Dubrovin valuation ring. □

We denote by $F(R), F(R_R)$ and $F(_RR)$ the set of all R-ideals in $Q(R)$, the set of all right R-ideals in $Q(R)$ and the set of all left R-ideals in $Q(R)$, respectively. We similarly use the notation $F(S), F(S_S)$ and $F(_SS)$ for the ring S.

Recall that a right S-submodule V' of $Q(V)$ is called a *right S-submodule* in $Q(V)$ if there is a regular element c in R such that $cV' \subseteq V$ and $V'Q(S) = Q(V)$. Similarly, we define a left R-submodule in $Q(V)$ (see [8]).

We denote by $F(V_S)$ the set of all right S-submodules in $Q(V)$, $F(_RV)$ the set of all left R-submodules in $Q(V)$ and put $F(_RV_S) = F(V_S) \cap F(_RV)$, that is, the set of all (R, S)-submodules in $Q(V)$.

We similarly denote by $F(_SW)$ the set of all left S-submodules in $Q(W)$, $F(W_R)$ the set of all right R-submodules in $Q(W)$ and $F(_SW_R) = F(_SW) \cap F(W_R)$.

Lemma 2.5. *Suppose that $R = VW$ and $S = WV$. Then the mappings*

$$I \to IV, \quad V' \to V'W,$$

where $I \in F(R_R)$ and $V' \in F(V_S)$, are order preserving one-to-one correspondences between $F(R_R)$ and $F(V_S)$. In particular, $I \in F(R)$ if and only if $IV \in F(_RV_S)$.

Proof. Let $I \in F(R_R)$. Then there is a regular element $c \in R$ such that $cI \subseteq R$ and so $cIV \subseteq RV = V$. Further $IVQ(S) = IQ(R)V = Q(R)V = Q(V)$. Hence $IV \in F(V_S)$. Conversely let $V' \in F(V_S)$. Then there is a regular element $d \in R$ such that $dV' \subseteq V$ and so $dV'W \subseteq VW = R$. Further $V'WQ(R) = V'Q(S)W = Q(V)W = Q(R)VW = Q(R)$ and so $V'W$ contains a regular element in R, that is, $V'W \in F(R_R)$. Since $R = VW$ and $S = WV$, the proof of the lemma is now trivial. □

In case $R = VW$ and $S = WV$, we have the following order preserving one-to-one correspondences.

Remark 2.2. (1) The mappings: $F(_RR) \to F(_SW)$ given by $I \to WI$ and $W' \to VW'$, where $I \in F(_RR)$ and $W' \in F(_SW)$. $I \in F(R)$ if and only if $WI \in F(_SW_R)$.

(2) The mappings: $F(S_S) \to F(W_R)$ given by $J \to JW$, $W' \to W'V$, where $J \in F(S_S)$ and $W' \in F(W_R)$. $J \in F(S)$ if and only if $JW \in F(_SW_R)$.

(3) The mappings: $F({}_SS) \to F({}_RV)$ given by $J \to VJ$ and $V' \to WV'$, where $J \in F({}_SS)$ and $V' \in F({}_RV)$. $J \in F(S)$ if and only if $VJ \in F({}_RV_S)$.

Definition 2.1. V is called an (R, S)-Dubrovin valuation bimodule in $Q(V)$ with V_1 if there is an (R, S)-submodule V_1 of V such that

(1) $\bar{V} = V/V_1$ is (R, S)-simple, ${}_R\bar{V}$ is left Artinian and $\bar{V}_S$ is right Artinian and
(2) For $\tilde{v} \in Q(V)\backslash V$ there are $r \in R$ and $s \in S$ with $r\tilde{v} \in V\backslash V_1$ and $\tilde{v}s \in V\backslash V_1$.

Similarly, we define an (S, R)-Dubrovin valuation bimodule in $Q(W)$ with W_1.

Lemma 2.6. *Suppose that $R = VW$ and $S = WV$. Let V_1 be an (R, S)-submodule of V and W_1 be an (S, R)-submodule of W such that $WV_1 = W_1V$. Put $M = V_1W$ and $N = W_1V$. Then*

(1) *$\bar{R} = R/M$ is a simple Artinian ring if and only if $\bar{V} = V/V_1$ is (R, S)-simple, $\bar{V}_S$ is right Artinian and ${}_S\bar{W} = W/W_1$ is left Artinian.*
(2) *$\bar{S} = S/N$ is a simple Artinian ring if and only if $\bar{W}$ is (S, R)-simple, $\bar{W}_R$ is right Artinian and ${}_R\bar{V}$ is left Artinian.*

Proof. It is clear that M and N are nonzero ideals of R and of S, respectively.

(1) By Lemma 2.5 $\bar{R}$ is a simple ring if and only if $\bar{V}$ is (R, S)-simple. It follows that $VW_1 = VW_1VW = VWV_1W = V_1W = M$. Hence $\bar{R}$ is a simple Artinian ring if and only if $\bar{V}_S$ is right Artinian and ${}_S\bar{W}$ is left Artinian by Lemma 2.5 and Remark 2.2(1).
(2) Follows from (2) and (3) in Remark 2.2 since $N = W_1V = WV_1$. □

Remark 2.3. Under the conditions $R = VW$ and $S = WV$, let $V_1 = V_0$ and $W_1 = W_0$. Then by Lemma 2.2 we have: $WV_0 = WVJ(S) = J(S) = J(S)WV = W_0V$, $M = J(R)$ and $N = J(S)$.

Theorem 2.1. *The following conditions on $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ are equivalent.*

(1) *T is a Dubrovin valuation ring.*
(2) (i) *R and S are Dubrovin valuation rings and*
 (ii) *$R = VW$ and $S = WV$.*
(3) (i) *V is an (R, S)-Dubrovin valuation bimodule in $Q(V)$ with V_1 and W is an (S, R)-Dubrovin valuation bimodule in $Q(W)$ with W_1 such that $WV_1 = W_1V$ and*
 (ii) *$R = VW$ and $S = WV$.*

Proof. (1) ⇒ (2): This follows from Lemma 2.4.

(2) ⇒ (1): $\bar{T}$ is a simple Artinian ring by Lemmas 2.3, 2.6 and Remark 2.3. Let $\alpha = \begin{pmatrix} \tilde{r} & \tilde{v} \\ \tilde{w} & \tilde{s} \end{pmatrix} \in Q(T)\backslash T$. We claim that there is a $\beta \in T$ with $\alpha\beta \in T\backslash J(T)$ as follows.

(a) If $\tilde{r} \notin R$, then there is an $r \in R$ such that $\tilde{r}r \in R\backslash J(R)$ and $\alpha\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{r}r & 0 \\ \tilde{w}r & 0 \end{pmatrix}$. If $\tilde{w}r \in W$, then $\alpha\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in T\backslash J(T)$ and $\beta = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in T$ as desired. In case $\tilde{w}r \notin W$. Since $S = WV$, we write: $1 = \sum_{i=1}^{n} w_i v_i$ for some $w_i \in W$ and $v_i \in V$. If $v_i\tilde{w}r \in R$ for all i, then $\tilde{w}r \in W$, a contradiction. So we may assume that $v_i\tilde{w}r \notin R(1 \leq i \leq j)$ for some j and $v_k\tilde{w}r \in R(j < k \leq n)$. Then there is an $r_1 \in R$ such that $v_1\tilde{w}rr_1 \in R\backslash J(R)$. If $v_i\tilde{w}rr_1 \in R$ for all $i(2 \leq i \leq j)$, then $\tilde{w}rr_1 \in W$ and $\tilde{w}rr_1 \notin W_0$, it is because if $\tilde{w}rr_1 \in W_0 = WJ(R)$ (Lemma 2.2), then $v_1\tilde{w}rr_1 \in VWJ(R) = J(R)$, a contradiction. Hence $\alpha\begin{pmatrix} rr_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{r}rr_1 & 0 \\ \tilde{w}rr_1 & 0 \end{pmatrix} \in T\backslash J(T)$ and $\beta = \begin{pmatrix} rr_1 & 0 \\ 0 & 0 \end{pmatrix} \in T$ as desired. If there is an i $(2 \leq i \leq j)$, say $i = 2$ with $v_2\tilde{w}rr_1 \notin R$, then there is an $r_2 \in R$ with $v_2\tilde{w}rr_1r_2 \in R\backslash J(R)$. We continue the same process and we have; for some $k(2 < k \leq j)$ such that $\alpha\begin{pmatrix} rr_1r_2\cdots r_k & 0 \\ 0 & 0 \end{pmatrix} \in T\backslash J(T)$ and $\beta = \begin{pmatrix} rr_1r_2\cdots r_k & 0 \\ 0 & 0 \end{pmatrix} \in T$ as desired.

(b) If $\tilde{s} \notin S$, then as in (a) but using $R = VW$ instead of $S = WV$ we have $\alpha\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \in T\backslash J(T)$ for some $s \in S$ and $\beta = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \in T$.

(c) If $\tilde{w} \notin W$, then there is an i with $v_i\tilde{w} \notin R$ as before, say $i = 1$. There is an $r_1 \in R$ such that $v_1\tilde{w}r_1 \in R\backslash J(R)$. If $v_i\tilde{w}r_1 \in R$ for all $i(2 \leq i \leq n)$, then $\tilde{w}r_1 \in W$ and $\tilde{w}r_1 \notin W_0$, it is because if $\tilde{w}r_1 \in W_0 = WJ(R)$, then $v_1\tilde{w}r_1 \in VW_0 = VWJ(R) = J(R)$, a contradiction. Hence if $\tilde{r}r_1 \in R$, then $\alpha\begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{r}r_1 & 0 \\ \tilde{w}r_1 & 0 \end{pmatrix} \in T\backslash J(T)$ and $\beta = \begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix} \in T$. If $\tilde{r}r_1 \notin R$, then we can go back to (a).

(d) If $\tilde{v} \notin V$, then we have $\alpha\beta \in T\backslash J(T)$ for some $\beta \in T$ as in (c) by using $R = VW$.

Similarly, there is a $\gamma \in T$ such that $\gamma\alpha \in T\backslash J(T)$ and hence T is a Dubrovin valuation ring.

(2) $\Rightarrow$ (3): To prove that V is an (R, S)-Dubrovin valuation bimodule in $Q(V)$ and W is an (R, S)-Dubrovin valuation bimodule in $Q(W)$, we take V_0 and W_0 as an (R, S)-submodule V_1 of V and as an (S, R)-submodule W_1 of W, respectively. By Lemma 2.6 and Remark 2.3, $\bar{V} = V/V_0$ is (R, S)-simple, ${}_R\bar{V}$ is left Artinian and $\bar{V}_S$ is right Artinian.

Let $\tilde{v} \in Q(V)\backslash V$. Then $\tilde{v}W$ is not contained in R and so there is a $w \in W$ with $\tilde{v}w \notin R$. Thus $\tilde{v}wr \in R\backslash J(R)$ for some $r \in R$ and $\tilde{v}wrV \subseteq V$ follows. If $\tilde{v}wrV \subseteq V_0 = J(R)V$, then $\tilde{v}wr \in \tilde{v}wrVW \subseteq J(R)VW = J(R)$, a contradiction and hence $\tilde{v}wrv_0 \in V\backslash V_0$ for some $v_0 \in V$, that is, $\tilde{v}s \in V\backslash V_0$, where $s = wrv_0 \in S$. Similarly, there is an $r \in R$ such that $r\tilde{v} \in V\backslash J(V)$. Hence V is an (R, S)-Dubrovin valuation bimodule in $Q(V)$ with V_0 by Lemma 2.6. Similarly, W is an (S, R)-Dubrovin valuation bimodule in $Q(W)$ with W_0.

(3) $\Rightarrow$ (2): Put $M = V_1W$ and $\bar{R} = R/M$. Then $\bar{R}$ is a simple Artinian ring. by Lemma 2.6. Let $q \in Q(R)\backslash R$. Then $qV \nsubseteq V$ and so there is a $v \in V$ with

$qv \notin V$. So $qvs \in V \backslash V_1$ for some $s \in S$ and $qvsW \subseteq VW = R$. If $qvsW \subseteq M$, then $qvs \in qvsWV \subseteq MV = V_1WV = V_1$, a contradiction. Hence there is a $w \in W$ such that $qvsw \in R \backslash M$ and $vsw \in R$. The symmetric argument shows that there is an $r \in R$ with $rq \in R \backslash M$ and hence R is a Dubrovin valuation ring. Similarly, S is a Dubrovin valuation ring. This completes the proof. □

Remark 2.4. The condition (2)(ii) in Theorem 2.1 is necessary for T to be a Dubrovin valuation ring as follows: For a Dubrovin valuation ring R with $J(R)^2 = J(R)$, let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ with $V = J(R) = W$ and $S = R$. Then $VW = J(R) \neq R$.

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