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RESEARCH ARTICLE | JUNE 28 2023

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AIP Conference Proceedings 2733, 030001 (2023)
https://doi.org/10.1063/5.0140151

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On a Truss-Module

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Abstract. A ring is one of the most essential structures in abstract algebra. There exist rings in the evolution of abstract algebra that contain "abnormal" members, particularly nilpotent elements. The existence of radicals of rings was inspired by this. Radical Jacobson is one of the most popular radical rings. Rump presented braces in 2007, and interestingly, the Jacobson radical $\mathscr{J}(A)$ of every ring, A, is a two-sided brace. Furthermore, in 2017, Brzeziski developed trusses, a novel construction that sits between the brace and the rings. In this research, we implement a qualitative literature study method to observe some fundamental properties of braces and trusses. Finally, as the result of this paper, we give some examples of trusses and show that every truss is a truss-module.

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INTRODUCTION

We believe we are all familiar with the following definition of ring. The following definition of ring is likely to be familiar to everyone. If $(R,+)$ forms a group and $(R,.)$ forms a semigroup and fulfills distributive laws, a non-empty set R with two binary operations $(+)$ and $(.)$ is termed a ring $(R,+,.)$, both left and right. The ring R will be referred to as a ring with identity if identity is present. Furthermore, if $a,b \in R$, implies $ab = ba$, the ring R is then referred to be a commutative ring. The sets $\mathbb{Z}$ and $\mathbb{Z}_n$ are (easy) examples or rings

The existence of ring has prompted a generalization of vector space as taught in linear algebra. A module is the name for this type of generalization. The set of all real numbers is a field in a vector space that performs scalar multiplication on the vectors, subject to certain axioms like the distributive property. The module notion is a significant generalization because the scalars in a module only have to be a ring. Ideals and quotient rings are both modules in commutative algebra, therefore many arguments about ideals or quotient rings can be integrated into a single argument about modules. Despite the fact that various ring-theoretic conditions can be written in non-commutative algebra for either left ideals or left modules, the distinction between left ideals, ideals, and modules becomes increasingly clear. Assume R represents a ring. A left $R-$module M is made up of an abelian group $(M,+)$ and an operation : $R \times M \to M$ that satisfies the following requirements for every $r,s \in R$ and $x,y \in M$:

1. $r \cdot (x+y) = r \cdot x + r \cdot y$
2. $(r+s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$

In the case of R having the identity 1 and $1 \cdot x = x$, then M is a unitary [1]. The Yang–Baxter equation (also known as the star–triangle link), on the other hand, is a physics consistency equation that was first used to statistical mechanics. It states that a matrix $\check{A}$ acting on two out of three objects satisfies the condition [2]:

$$(\check{A} \otimes \mathbf{1})(\mathbf{1} \otimes \check{A})(\check{A} \otimes \mathbf{1}) = (\mathbf{1} \otimes \check{A})(\check{A} \otimes \mathbf{1})(\mathbf{1} \otimes \check{A}) \quad (1)$$

It gets its name from the 1968 and 1971 separate works of C. N. Yang and R. J. Baxter. The Yang-Baxter equation's importance in creating knot invariants using the statistical mechanics method is reexamined and clarified. With the inclusion of piecewise-linear lattices and upgraded vertex and interaction-round-a-face (IRF) models with strictly local weights, the definition of knot invariants becomes more exact. The Yang-Baxter equation and the obligation to

Proceedings of the 3rd Ahmad Dahlan International Conference on Mathematics and Mathematics Education 2021
AIP Conf. Proc. 2733, 030001-1–030001-6; https://doi.org/10.1063/5.0140151
Published by AIP Publishing. 978-0-7354-4571-0/$30.00

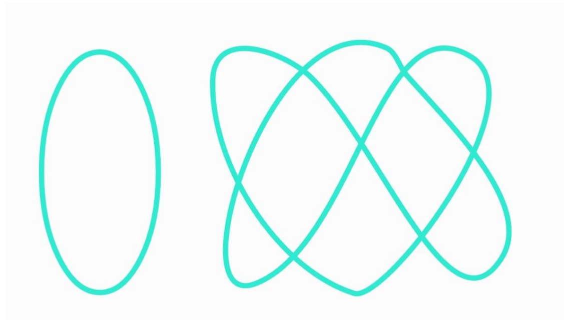

FIGURE 1. Unknot (left), knot (right).

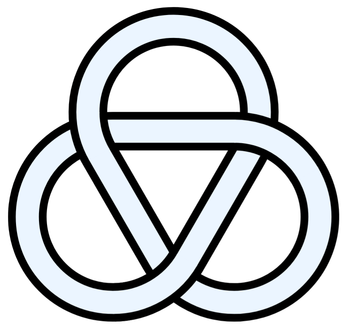

FIGURE 2. Trefoil Knot in 2-Dimensional Shape.

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utilize its solution in the infinite speed limit naturally arise in the implementation of invariances under Reidemeister movements III. It's also shown that charge-conserving vertex models are required, and that producing knot invariants from IRF models follows directly from the vertex model formulation [3].

In the mathematical discipline of topology, knot theory is the study of mathematical knots. While mathematical knots are inspired by everyday knots like those seen in shoelaces and rope, they differ in that the ends are linked to prevent them from being undone, with the simplest knot being a ring (or "unknot"). In mathematical jargon, a knot is a 3-dimensional Euclidean space $\mathbb{R}^3$ embedding of a circle. A circle is tied to all of its homeomorphisms in topology, not only the conventional geometric concept. In [4], some knot properties in physics were studied. Knot and link invariants are generalized amplitudes for a quasi-physical process, according to Kauffman [4]. According to historians, the knot hypothesis began in Ancient Greece. Although, we don't have any proof that knot theory was regarded a branch of mathematics by ancient Greeks, but surgeons surely did. An illustration of unknot and knot are given in the Figure 1.

One of the most famous of knots is trefoil. In virtue of the trefoil knot property, the trefoil knot is the curve generated by solving the following parametric equations [5]:

$$x = \sin(t) + 2\sin(2t) \tag{2}$$

$$y = \cos(t) - 2\cos(2t) \tag{3}$$

$$z = -\sin(3t) \tag{4}$$

A simple illustration of the trefoil knot can be seen in Figure 2 and Figure 3.
Furthermore, the trefoil knot has some representations, namely.

1. The trefoil knot's Alexander polynomial is $\Delta(t) = t - 1 + t^{-1}$ which is illustrated in the Figure 4.

2. The trefoil's Kauffman polynomial is $L(a,z) = za^5 + z^2a^4 - a^4 + za^3 + z^2a^2 - 2a^2$ which is illustrated in the Figure 5.

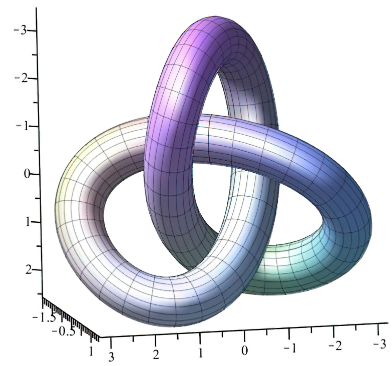

FIGURE 3. Trefoil Knot in 3-Dimensional Shape.

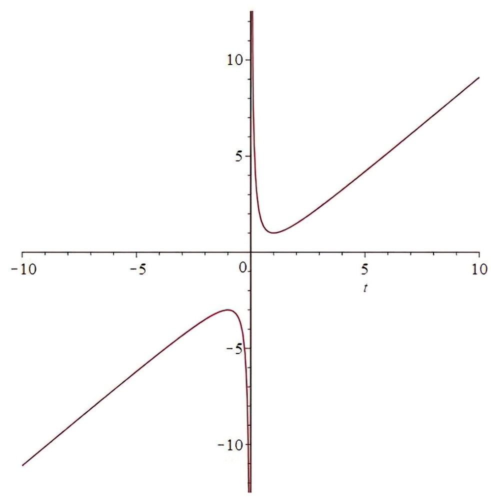

FIGURE 4. Alexander polynomial of the trefoil knot using MAPLE.

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BRACES AND TRUSSES

Rump suggested braces as a generalization of Jacobson radical rings in 2007 for exploring Yang–Baxter problem involutive non-degenerate set-theoretic solutions, a basic equation in Mathematics and physics [6, 7]. Remember that a left brace is a set B joint with two binary operations $+$ and $(.)$, such that $(B,+)$ is an abelian group, $(B,.)$ is a group, and for all $a,b,c \in B$ fulfill these conditions:

$$a(b+c)+a = ab+ac \tag{5}$$

If the equality 5 can be replaced by 6, then B is a right brace [8].

$$(b+c)a+a = ba+ca \tag{6}$$

Some further results on braces can be found in [9, 10, 11]. If a brace is both left and right, then it will be called a two-sided brace. The Jacobson radical $\mathscr{J}(A)$ of any ring A is a two-sided brace. In general, every member of the Jacobson radical class $\mathscr{J}$ is a two-sided brace. A concrete example of a Jacobson ring can be seen in [12].

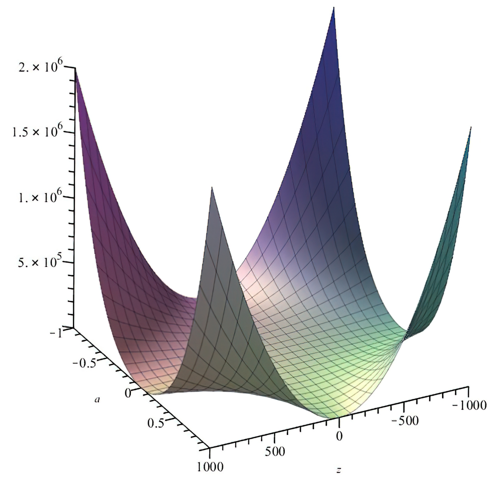

FIGURE 5. Kauffman polynomial of the trefoil knot using MAPLE.

On the other hand, in 2019, Brzezinski introduced a truss. Recall the definition of heard. A nonempty set D with a ternary operation $[-,-,-]: D\times D\times D\to D$ is called a heard if for all $a,b,c,d,e\in D$ satisfy

$$[[a,b,c],d,e]=[a,b,[c,d,e]],\quad [a,b,b]=a=[b,b,a] \tag{7}$$

Moreover, if $[a,b,c]=[c,b,a],\forall a,b,c\in D$, then D is called an abelian heard [13].

A truss is an Abelian heard R combined with an associative binary operation that distributes over the heard operation, i.e., for all $a,b,b',b"\in R$ satisfy [13]:

$$a[b,b',b"]=[ab,ab',ab"]\quad [b,b',b"]a=[ba,b'a,b"a] \tag{8}$$

In his article [13], Brzezinski introduced a truss-module. In the next section, we'll look at a truss-module and show that every truss is a truss-module.

RESULT AND DISCUSSION

This section begins with some specific instances of trusses.

Example 1. *The set $\mathbb{Z}$ of all integers is a truss.*

Proof. Now, we define $[-,-,-]:\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ as $[x,y,z]=x-y+z,\forall x,y,z\in\mathbb{Z}$. Let $a,b,c,d,e\in\mathbb{Z}$. Consequently, we have

$$[[a,b,c],d,e]=[a,b,c]-d+e=a-b+c-d+e \tag{9}$$

$$[a,[d,c,b],e]=a-[d,c,b]+c=a-d+c-b+e \tag{10}$$

$$[a,b,[c,d,e]]=a-b+[c,d,e]=a-b+c-d+e \tag{11}$$

It follows from the Equation 9, Equation 10, and Equation 11 that

$$[[a,b,c],d,e]=[a,[d,c,b],e]=[a,b,[c,d,e]] \tag{12}$$

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Moreover, $[a,b,c] = a-b+c = c-b+a = [c,b,a]$. Therefore, $\mathbb{Z}$ with the ternary operation $[x,y,z] = x-y+z, \forall x,y,z \in \mathbb{Z}$ is an abelian heard.

Furthermore, we have

$$a[b,c,d] = a(b-c+d) = ab-ac+ad = [ab,ac,ad] \tag{13}$$

$$[b,c,d]a = (b-c+d)a = ba-ca+da = [ba,ca,da] \tag{14}$$

It follows from Equation 13 and 14 that $\mathbb{Z}$ with the ternary operation $[x,y,z] = x-y+z, \forall x,y,z \in \mathbb{Z}$ is a truss. □

Example 2. *The set*

$$M_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{R} \}$$

of all $2 \times 2-$*matrices over* $\mathbb{R}$ *is a truss.*

Proof. The set $M_2(\mathbb{R})$ is a truss with the following ternary operation.

$$[-,-,-] : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

defined as $[A,B,C] = A-B+C$ for every $A,B,C \in M_2(\mathbb{R})$. □

In general, we've got

Lemma 3. *Every ring is a truss*

Proof. Let R be a ring with the binary operation $+$ and $\times$. We define

$$[-,-,-] : R \times R \times R \to R$$

as $[p,q,r] = p-q+r, \forall p,q,r \in R$. This ternary operation on R exists since $\forall q \in R, \exists -q \in R$. Thus R is a truss and it is denoted by $T(R)$ □

The converse of Lemma 3, on the other hand, is not true in the following case.

Example 4. *It is clear that the set* $O(\mathbb{Z})$ *of all odd integers is not a ring, but it is a truss since the ternary operation* $[a,b,c] = a-b+c$ *maps* $O(\mathbb{Z}) \times O(\mathbb{Z}) \times O(\mathbb{Z}) \to O(\mathbb{Z})$*, where* $+$ *is just integer addition together with multiplication of integers is also closed on* $O(\mathbb{Z})$.

Now, recall the definition of a truss-module.

Definition 5. *[13] Let R be a truss and let N be an abelian heard. The abelian heard N. The abelian heard N is a left* $R-$*module if there exists an associative left action* $\alpha_N : R \times N \to N$ *of R on N that distributes over a heard operation such that* $\forall r, r', r" \in R$ *and* $\forall n, n', n" \in N$ *satisfy*

1. $r(r'n) = (rr')n$
2. $[r,r',r"]n = [rn, r'n, r"n]$
3. $r[n,n',n"] = [rn, rn', rn"]$

If R is a truss with identity and satisfies $1.n = n$ *for every* $n \in N$*, then we say that N is a unitary truss-module.*

In virtue of Definition , we directly have the following property.

Theorem 6. *Every truss is a truss-module over itself.*

Proof. Let T be a truss. It is clear that T is an abelian heard. Now let $r, r', r", n, n', n" \in T$ so that

1. $r(r'n) = (rr')n$ since the associativity holds on T,
2. $[r,r',r"]n = (r-r'+r")n = rn - r'n + r"n = [rn, r'n, r"n]$,

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3. $r[n,n',n"] = r(n-n'+n") = rn - rn' + rn" = [rn,rn',rn"]$.

Hence, T is a $T-$truss-module. □

Example 7. *The following examples are truss-modules*

1. *Let M be an $R-$module. Then, M is an $T(R)-$truss-module.*
2. *Now consider the set $O(\mathbb{Z})$ of all odd integers and consider the truss $T(2\mathbb{Z})$. The set $O(\mathbb{Z}) \times T(2\mathbb{Z}) = \{(m,n)|m \in O(\mathbb{Z}), m \in T(2\mathbb{Z})\}$ together with component-wise addition and multiplication forms a truss. Thus $O(\mathbb{Z}) \times T(2\mathbb{Z})$ is an abelian heard. Furthermore, it is easy to check that $O(\mathbb{Z}) \times T(2\mathbb{Z})$ is a $O(\mathbb{Z}) \times T(2\mathbb{Z})-$truss module.*

Assume T is a truss. If S is a sub-heard of T and $as, sa \in S, \forall s \in S, a \in T$, a non-empty subset S is termed a truss-ideal of T. [14]. Let M be a left $T-$truss-module. A subheard N of M is called a submodule of M if it is closed under $T-$action [13]. In the study of truss, to have algebraic structure associated with any congruence we need to consider paragons, they were introduced by Tomasz Brzezinski in his paper [15].

CONCLUSION

Due to the construction of a truss, there are various chances to investigate the properties of trusses and truss-modules. Some questions, such as how to define a prime truss-module and how to define the radical of a truss or truss-module, should be investigated further in future research.

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ACKNOWLEDGMENTS

The authors express their gratitude to everyone who helped them with this study. This research was done in collaboration with the Mathematics Education Department, Universitas Ahmad Dahlan, and Bina Nusantara University. The first author also express his gratitude to Bernard Rybołowicz, Heriot-Watt University, United Kingdom, for giving advice and some beautiful examples of trusses.

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