THE WAVELET PROJECTION METHOD FOR SOLVING OPERATOR EQUATIONS

THESIS SUMMARY

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1 Introduction

We have three reasons to decide the topic of the research :

1. The fact that wavelet has many advantages properties, such as locality, orthogonality, stability, regularity, vanishing moment property, and fast algorithms so that in recent years the application of wavelet in the science and engineering, in particular in the signal processing is growing rapidly. (Mallat 1998, Chui 1997, Tang et al., 2000, Kobayashi at al., 1998).

2. The operator equations as mathematical model are much needed in the science and technology. The final goal on the mathematical model is to obtain the solution.

3. Although there are available a number of methods and tools to determine the solution of operator equations, we still find some limitations. Those limitations are caused by the more requirement for quality of approximation. Also, the problems which should be solved is broader and more complex.

Generally, the obstacles in applying the wavelet in this field relate to the following facts :

1. The wavelet with the advantage properties is defined on whole real line $\mathbb{R}$, while the operator equations to be defined only on a finite interval or domain.

2. Usually, the wavelet is very good for dealing the data which are available explicitly, but the solution of the operator equations is only available implicitly.

In this research we are concerned with the following problems :

1. To develop the approximation schemes the application of the periodized wavelets basis for non periodic boundary value problems (BVP). The motivation is to extend the application of the periodic wavelet because so far this basis only be used to the periodic BVP (Nielson, 1998; Kucera & Vijek, 1999 dan Rieder, 2000).

2. To develop the approximation schemes based on the collocation method so that we get the simpler methods to handle various kinds of operator equations. This idea is inspirited by difficulty to do the numerical realization of the method using the wavelets (Jaffard, 1992; Dahmen at al., 1999; Barinka at al., 1999; Hernadi and Soedijono, 2000; Urban, 2002), and the difficulty is caused by unavailable the orthogonal wavelets basis on the interval.

3. To study the Riesz basis properties of the restricted wavelets basis arising a non orthonormal wavelet. The motivation is to keep a stable representation the solution in the restricted wavelet basis, to weaken the orthonormal condition of the restricted wavelet basis by Jaffard and Laurencort (1992) as well as to make easily for choosing the wavelet basis because there are more alternatives.

4. To give the justification theoretically the approximation schemes using the wavelet. The results concerning with the uniquely approximation-solvable by Zeidler (1990) are developed so that the wavelets basis is applicable.

5. To build the approximation schemes for the time-dependent problems and the integral equation. The approach is different from the schemes introduced by Beylkin (1993), Sweldens (1994), Nielson (1998), Kucera dan Vicek (1999), Vasilyev dkk (1995), Cai dan Wang (1996), Hernadi (2002).

6. To solve some problems on wavelet computations for numerical realization. The techniques discovered by Sweldens (1994), Nielson (1998), Strang & Nguyen (1996), Barker (2001), are followed and developed. The computations are continued up to compose the algorithms and the MATLAB functions.

The common goal of this study is to participate actively in the scientific research so that the results give the valuable contributions for the development of sciences, especially mathematics. Meanwhile, the specific goal is to build some approximation schemes for solving the operator equation using the wavelet projection method.

Furthermore, we hope the results of the research will be useful for the people who use the mathematical models, in particular to obtain the solution of the operator equations. Besides that, the research would extend the application of wavelet on the numerical analysis. The open problems appearing in the research will give the chance and the challenge to do the further research on the area of wavelet, operator equations and numerical analysis.

2 Preliminaries

The preliminaries consist of some topics from functional analysis, especially the projection and the operator equation, the basic concept of wavelet and some definitions from numerical analysis. By reason of simplicity, we only present the preliminaries on the wavelet theory. The others are assumed to be understood. Following, the preliminaries of wavelets summarized from (Mallat, 1989; Daubechies, 1992; Chui, 1992; Eirola (1992); Jaffard and Laurencort, 1992, Nielson, 1998).

There are two ways to introduce the wavelet, through the continuous wavelet transform (CWT) and multiresolution analysis (MRA). Wavelet is a function $\psi \in \mathcal{L}^2(\mathbb{R})$ which satisfies the admissibility condition, i.e.

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty.$$

where $\hat{\psi}$ is the Fourier transform of ψ. This condition requires ψ fulfils

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x)dx = 0.$$

On the other side, the wavelet ψ is also derived from MRA. Multiresolution analysis is a collection of closed subspaces $\{V_j\}_{j\in Z} \in \mathcal{L}^2(\mathbb{R})$ with the following properties :

(i) $\{0\} \subset ... \subset V_{-1} \subset V_0 \subset V_1 ... \subset \mathcal{L}^2(\mathbb{R})$

(ii) $\overline{\bigcup_{j=\infty}^{\infty} V_j} = \mathcal{L}^2(\mathbb{R})$ and $\bigcap_{j=\infty}^{\infty} V_j = \{0\}$

(iii) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$

(iv) there exists $\phi \in \mathcal{L}^2(\mathbb{R})$ such that $\{\phi(x-k)\}_{k\in Z}$ constitutes an orthonormal basis of V_0.

The function ϕ on (iv) is called the scaling function or father wavelet. The scaling function ϕ has the unit area property, i.e.

$$\int_{-\infty}^{\infty} \phi(x)dx = 1.$$

If $P_j : \mathcal{L}^2(\mathbb{R}) \to V_j$ is an orthogonal projection then $\lim_{j\to\infty} P_j f = f$ in $\mathcal{L}^2(\mathbb{R})$ for any $f \in \mathcal{L}^2(\mathbb{R})$. This means any function in $\mathcal{L}^2(\mathbb{R})$ can be approximated accurately by a function in a space of MRA.

Given $\{V_j\}_{j=\infty}^{\infty}$ any MRA of $\mathcal{L}^2(\mathbb{R})$. For each $j \in Z$ define W_j as the orthogonal complement of V_j in V_{j+1}, i.e.

$$V_{j+1} = V_j \oplus W_j. \qquad (2.1)$$

The space of such W_j is called the detail space. If there is an orthonormal scaling function $\phi \in \mathcal{L}^2(\mathbb{R})$ then there always exists a function $\psi \in \mathcal{L}^2(\mathbb{R})$ such that $\{\psi(x-k)\}_{k\in Z}$ is an orthonormal basis of W_0. This function ψ is called the wavelet or mother wavelet. The wavelets basis consist of the scaling and dilating version of wavelet $\phi_{j,k}$, $\psi_{j,k}$ where

$$\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k) \text{ dan } \psi_{j,k}(x) := 2^{j/2}\phi(2^j x - k), j, k \in Z.$$

The set $\{\phi(x-k)\}_{k\in Z}$ is called the Riesz basis for V_0 if and only if there are two constants A and B with $0 < A \leq B < \infty$ such that for any sequence $(c_k) \in \ell^2(Z)$ we have

$$A\big\|\{c_k\}\big\|_{\ell_2} \leq \big\|\sum_{k\in Z} c_k\phi(x-k)\big\| \leq B\big\|\{c_k\}\big\|_{\ell_2}. \tag{2.2}$$

The orthonormal basis is a special case where $A = B = 1$.

The local property of wavelet is characterized by its support. For the wavelet with support compact there are only finite number of non zero coefficients (a_k) such that ϕ satisfies two scales relation :

$$\phi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k\phi(2x-k). \tag{2.3}$$

Since $W_0 \subset V_1$ wavelet ψ can be obtained by

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k\phi(2x-k). \tag{2.4}$$

Here, the positive number D corresponds to width of the support and usually indicates the wavelet genus. In many cases, the coefficients (b_k) in (2.4) could be derived directly from (2.3) by the relation

$$b_k = (-1)^k a_{D-1-k}\ ,\ k = 0, 1, \ldots, D-1$$

Two families of wavelets which will be used are Daubechies and B-spline wavelet. Daubechies wavelets are characterized by an even number D, where for each even D there exists the scaling function ϕ and wavelet ψ satisfying (2.3) and (2.4), respectively with the following properties :

(i) **supp** $\phi =$ **supp** $\psi = [0, D-1]$,

(ii) ϕ and ψ constitute the orthonormal basis,

(iii) ψ has $D/2$ number of vanishing moments, i.e.

$$\int_{-\infty}^{\infty} x^p\psi(x)dx = 0\ ,\ p = 0, 1, 2, \ldots D/2-1$$

(iv) For D large enough, $\phi, \psi \in C^{\mu D/2}$ with $\mu \approx 0.2075$.

The Haar wavelet is the Daubechies wavelet with $D = 2$. Except for Haar case, the Daubechies wavelet does not have the explicitly formulae. Some the basic result concerning with the regularity of Daubechies wavelet is presented in Table 2.1. The function spaces we mean here are following :

Table 2.1. Regularity of Daubechies wavelet where ϕ, $\psi \in C^{\alpha}(\mathbb{R})$ and ϕ, $\psi \in \mathcal{H}^{\beta}(\mathbb{R})$

D	2	4	6	8	10	12	14	16	18	20
α	-	0.550	1.088	1.618	1.596	1.888	2.158	2.415	2.661	2.902
β	0.5	1	1.415	1.775	2.096	2.388	2.658	2.914	3.161	3.402

$$C^{\alpha}(\mathbb{R}) := \left\{ f \in C^{n} | n = \alpha - \sigma \in \mathbf{N}, 0 \leq \sigma < 1, \sup_{x \neq y} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x-y|^{\sigma}} < \infty \right\}$$

and

$$\mathcal{H}^{\beta}(\mathbb{R}) := \left\{ f \in \mathcal{L}^2 | \ \|f\|_{\beta} := \|\hat{f}(\omega)(1+|\omega|^2)^{\beta/2}\|_{\mathcal{L}^2} < \infty \right\}$$

The B-spline functions are characterized by a positive integer N (not need even), where for each integer N the B-spline function ϕ_N is defined iteratively as follows :

$$\phi_N = \phi_0 * \phi_{N-1} = \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{N+1 \text{ terms}} \qquad (2.5)$$

where $\phi_0 = \chi_{[0,1]}$ is the characteristic function on $[0,1]$ and the symbol $*$ stands for convolution operator. For each N, the B-spline function ϕ_N is a piecewise polynomial of degree N. In fact, in the context of wavelet the B-spline functions hold the conditions of the scaling function. Some properties of B-spline scaling function :

(i) $\mathbf{supp}\phi_N = [0, N+1]$ dan $\phi_N \in \mathcal{C}^{N-1}(\mathbb{R})$,

(ii) $\phi_N|_{[k,k+1]} \in \mathcal{P}_N$, $k = 0, \pm 1, \pm 2, \cdots$,

(iii) $\phi_N(x) > 0$ untuk $0 < x < N+1$,

(iv) $\sum_{k \in Z} \phi_N(x-k) = 1, x \in \mathbb{R}$,

(v) $\int_{-\infty}^{\infty} \phi_N(x)dx = 1$,

(vi) $\phi'_N(x) = \phi_{N-1}(x) - \phi_{N-1}(x-1)$,

(vii) ϕ_N can be calculated from ϕ_{N-1} by the identity,

$$\phi_N(x) = \frac{x}{N}\phi_{N-1}(x) + \frac{N+1-x}{N}\phi_{N-1}(x-1).$$

3 Some Properties of Wavelets Basis on the Interval

In order to deal with the operator equations which are defined on a bounded interval we use two adapted wavelet bases, namely the periodized and the restricted wavelets basis. Let ϕ, $\psi \in \mathcal{L}^2(\mathbb{R})$ be the scaling function of a MRA and wavelet, respectively. The periodized wavelets basis are the functions $\tilde{\phi}_{j,\ell}$ and $\tilde{\psi}_{j,\ell}$, $j, \ell \in Z$ which are defined as :

$$\tilde{\phi}_{j,\ell}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,\ell}(x+n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \phi(2^j(x+n)-\ell) \tag{3.1}$$

and similarly for $\tilde{\psi}_{j,\ell}$. If ϕ as well as ψ forms an orthonomal basis then so does the periodized wavelet. Actually, on each level j there are only 2^j distinct periodized wavelet basis $\tilde{\phi}_{j,\ell}$, i.e.

$$\tilde{\phi}_{j,\ell}\ ,\ \ell = 0, 1, 2, \cdots, 2^j - 1.$$

Moreover, for each $j \geq 0$, we define two classes of subspaces

$$\tilde{V}_j = \mathbf{span}\big\{\tilde{\phi}_{j,\ell}, 0 \leq x \leq 1\big\}_{\ell=0}^{2^j-1} \text{ dan } \tilde{W}_j = \mathbf{span}\big\{\tilde{\psi}_{j,\ell}, 0 \leq x \leq 1\big\}_{\ell=0}^{2^j-1}.$$

The properties of these subspaces are the following :

(i) $\tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots \subset \mathcal{L}^2[0,1]$,

(ii) $\overline{\bigcup_{j=0}^{\infty} \tilde{V}_j} = \mathcal{L}^2[0,1]$,

(iii) $\tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j$,

(iv) $\mathcal{L}^2[0,1] = \tilde{V}_{J_0} \oplus \left(\bigoplus_{j=J_0}^{\infty} \tilde{W}_j \right)$.

Since that, the periodized wavelet basis wavelet can be used to approximate any function in $\mathcal{L}^2[0,1]$.

The restricted wavelets basis are the functions $\bar{\phi}_{j,k}$ defined as :

$$\bar{\phi}_{j,k}(x) = \phi_{j,k}(x)|_{[0,1]} \tag{3.2}$$

i.e. the restriction of $\phi_{j,k}$ to $[0,1]$. Define

$$V_j^{[0,1]} := \text{the restriction of function in } V_j \text{ to } [0,1].$$

The set $\{\bar{\phi}_{j,k} : k \in Z\}$ constitutes the basis for $V_j^{[0,1]}$. Moreover, we define the set of indices

$$I = \big\{k \in Z :\ I_{j,k} \cap (0,1) \neq \emptyset\big\}.$$

This set is partitioned into three disjoint index sets as follows :

$$\begin{aligned} I_1 &= \{k \in I : 0 \in I^o_{j,k}\}, \\ I_2 &= \{k \in I : I^o_{j,k} \subset [0,1]\}, \\ I_3 &= \{k \in I : 1 \in I^o_{j,k}\}. \end{aligned}$$

where $I^o_{j,k}$ the interior of $I_{j,k} = \textbf{supp}\ \phi_{j,k}$. We call $\bar{\phi}_{j,k}$ as the left,the internal, and the right basis element if k contained in I_1, I_2, and I_3, respectively. Some results concerning with the properties of wavelet basis on the interval are the following :

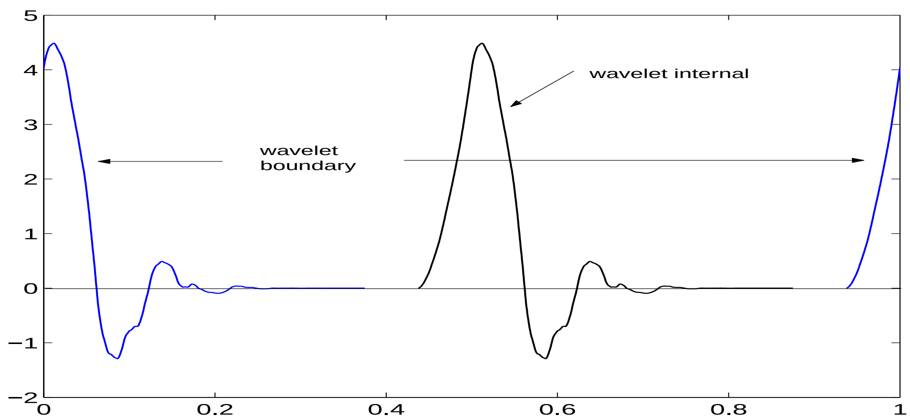

Figure 3.1: Graph of periodized wavelet on $[0,1]$.

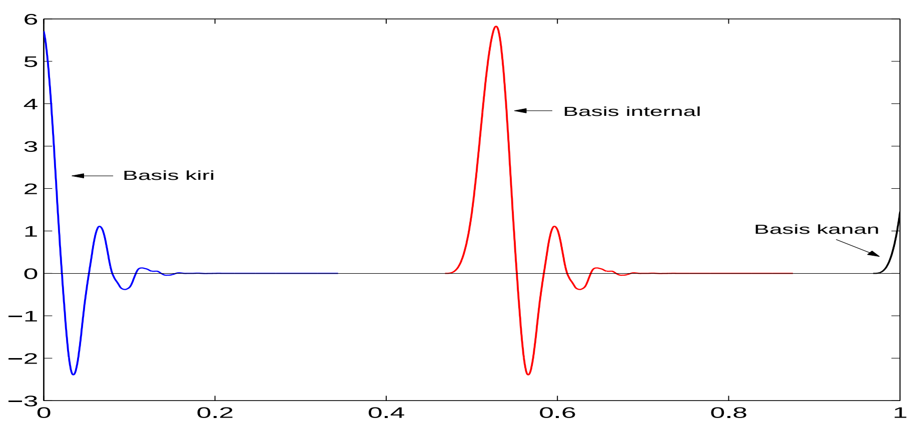

Figure 3.2: Graph of restricted wavelet

Theorem 3.1. *Let ψ be wavelet with* **supp** $\psi = [0, D-1]$. *If J_0 is some integer with $J_0 \geq {}^2\log(D-1)$ and $\tilde{\psi}_{j,\ell}$ are the periodic wavelets basis, then for each $j \geq J_0$ we have*

$$\tilde{\psi}_{j,\ell}(x) = \begin{cases} \psi_{j,\ell}(x) & \text{jika } x \in I_{j,\ell} \cap [0,1] \\ \psi_{j,\ell}(x+1) & \text{jika } x \notin I_{j,\ell}, x \in [0,1] \end{cases} \tag{3.3}$$

where $I_{j,\ell}$ is support of $\psi_{j,\ell}$. The similar statement for $\tilde{\phi}_{j,\ell}$.

Theorem 3.2. *Let $\{V_j\}_{j\in Z}$ be a MRA of $\mathcal{L}^2(\mathbb{R})$ which is generated by ϕ with* **supp** $\phi = [0, D-1]$. *If $\overline{\phi}_{j,k}$ are the restricted wavelets basis then for each $j \geq {}^2\log(2D-4)$, $\left\{\overline{\phi}_{j,k} : k \in I\right\}$ constitutes a Riesz basis for $V_j^{[0,1]}$. In addition, there are $2^j + D - 2$ number of restricted wavelet basis which consist of $D-2$ on the left, $2^j - D + 1$ in internal and $D-2$ on the right.*

Theorem 3.3. *Let both ϕ and ψ be a pairing of wavelets constitute the orthonormal wavelet bases,* **supp** $\phi =$ **supp** $\psi = [0, D-1]$. *If the following conditions holds :*

(i) ψ *has P number of vanishing moments,*

(ii) $\tilde{\phi}_{j,k}$ *and* $\tilde{\psi}_{j,k}$ *the periodized wavelets bases*

(iii) $f \in \mathcal{C}^P[0,1]$ *has a periodic extension* $\tilde{f} \in \mathcal{C}^P(\mathbb{R})$

then for each $J \geq^2 \log(D-1)$ we have

$$\|\widetilde{P}_J f - f\|_{\mathcal{L}^2[0,1]} = \mathcal{O}(2^{-JP})$$

where $\widetilde{P}_J$ is the orthogonal projection onto $\tilde{V}_J$.

Theorem 3.4. *Let ϕ be scaling function which generates MRA $\{V_j\}$. If ϕ satisfies the Strang-Fix condition of order P, i.e.*

$$\left.\frac{d^p}{d\omega^p}\hat{\phi}(\omega)\right|_{\omega=2k\pi} = 0 \text{ for } p = 0, 1, 2, \cdots, P-1 \text{ and } k \neq 0$$

and $f \in \mathcal{L}^2[0,1]$ has an extension $\tilde{f} \in \mathcal{L}^2(\mathbb{R})$ with $\tilde{f} \in \mathcal{C}^P(\mathbb{R})$ then we have

$$\left\|P_J^{[0,1]} f - f\right\|_{\mathcal{L}^2[0,1]} = \mathcal{O}(2^{-JP})$$

where $P_J^{[0,1]}$ is a projection on $V_J^{[0,1]}$.

4 Existence of Solution Projection Equation

Given the operator equation :

$$\mathbf{A}u = f \tag{4.1}$$

where $\mathbf{A}$ some operator from a Banach V into another Banach W and $f \in W$. Generally, the true solution of operator equation is presence in the space of infinite dimensions. The ide on the projection methods the solution is approximated step wise by functions arising from a finite dimension space V_n which be called the approximation space. Let $\{\phi_k : k \in \Lambda_n\}$ be basis for V_n. The solution is approximation by some function of the form :

$$u_n = \sum_{k\in\Lambda_n} c_k\phi_k \in V_n. \tag{4.2}$$

Hereafter, this function is called ansatz. In order to determine the approximation, the equation (4.1) is to be projected into subspace V_n, and we obtain the projection equation :

$$\mathbf{P}_n\mathbf{A}u_n = P_nf. \tag{4.3}$$

where $\mathbf{P}_n : V \longrightarrow V_n$ a projection operator, i.e. $\mathbf{P}_n^2 = \mathbf{P}_n$. Substituting the ansatz (5.6) into (4.1) we obtain the residual $r_n = \mathbf{A}u_n - f$. The collocation and the Galerkin method are two most popular projection method.

Furthermore, let us consider the equation (4.1) and its approximation (4.3). Assume that :

(A1) V is a Banach space and V_n is a subspace of finite dimension of V where

$$V_n \subset V_{n+1} \text{ dan } \overline{\bigcup_{n=1}^{\infty} V_n} = V.$$

(A2) $\mathbf{P}_n : V \longrightarrow V_n$ is a projection operator, i.e. $\mathbf{P}_n^2 = \mathbf{P}_n$.

Equation (4.1) is said to be uniquely approximation-solvable if the following conditions holds (Zeidler, 1990) :

(i) The original equation (4.1) has a unique solution.

(ii) There exist $n_0 \in \mathbb{N}$ such that the equation (4.3) has a unique solution for each $n \geq n_0$.

(iii) The sequence of solutions u_n to (4.3) converges to the solution of (4.1).

In the following, we present some theorems concerning with the uniquely approximation-solvable.

Theorem 4.1. *Assume that (A1) and (A2) holds. If $\mathbf{P}_n$ is an orthogonal projection and the operator $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{I} - \mathbf{K}$ with $\mathbf{K}$ linear and contraction, i.e. $\|\mathbf{K}\| < 1$ then the equation (4.1) is uniquely approximation-solvable. In addition, we have the estimation :*

$$\|u - u_n\| \leq \frac{1}{1 - \|\mathbf{K}\|}\|u - \mathbf{P}_nu\|. \tag{4.4}$$

Theorem 4.1 requires the projection $\mathbf{P}_n$ to be orthogonal. The following theorem does not need the orthogonal condition.

Theorem 4.2. *Assume that (A1) and (A2) holds. The sufficient and necessary condition for the equation (4.1) to be uniquely approximation-solvable are the operator* $\mathbf{A}$ *can be represented as* $\mathbf{A} = \mathbf{I} - \mathbf{K}$ *with* $\mathbf{K}$ *compact, and the equation* $\mathbf{A}u = 0$ *has only a trivial solution. Moreover, we have the estimation*

$$\|u - u_n\| \leq M\|u - \mathbf{P}_n u\| \text{ for some constant } M. \tag{4.5}$$

Theorem 4.3. *Let* V *be a Hilbert space and assumption (A1)-(A2) are satisfied. If the operator* $\mathbf{A}$ *is linear, continuous and V-elliptic then the equation (4.1) uniquely approximation-solvable. Furthermore, we have the estimation*

$$\|u - u_n\| \leq \frac{1}{\alpha}\|Au_n - f\| \tag{4.6}$$

where $\alpha > 0$ *some elliptic constant.*

5 The Approximation Schemes

In order to realize numerically the approximation schemes we need to solve some problems on wavelet computation, such as evaluating the function values and its derivatives of the wavelets basis, calculating the connection coefficients, calculating the moments and approximating the integral containing the product of wavelet and any function.

5.1 Method for solving non periodic BVP using periodized wavelet

Consider the following non homogen BVP :

$$\begin{cases} -u''(x) + \alpha u(x) = f(x) & x \in (0,1) \\ u(0) = u_0 \ , \ u(1) = u_1 \end{cases} \tag{5.1}$$

where $\alpha \geq 0$ dan $f \in \mathcal{L}^2(0,1)$.

5.1.1 The embedding domain approach

The idea of this approach is to extend the BVP (5.1) into a new domain $[c,d] \supset [0,1]$ so that the restriction of the solution on $[0,1]$ is the solution of the BVP (5.1). Taking the space of test functions :

$$H^1_{per}(c,d) = \{v \in H^1(c,d) : v(c) = v(d), v'(c) = v'(d)\}.$$

and using the ansatz

$$u_j(x) = \sum_{k=0}^{2^j-1} c_{j,k}\tilde{\phi}_{j,k}(x) \tag{5.2}$$

we obtain the linear system of equation as :

$$\begin{pmatrix} \mathbf{M} & -\phi_a & -\phi_b \\ -\phi_a^T & \mathbf{0} & \mathbf{0} \\ -\phi_b^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \lambda_a \\ \lambda_b \end{pmatrix} = \begin{pmatrix} \mathbf{c}_f \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{5.3}$$

where

$$\begin{aligned} \mathbf{M} &= \mathbf{K} + \alpha\mathbf{I},\ \mathbf{I} \text{ identity matrix .} \\ (\mathbf{K})_{k,\ell} &= \int_c^d \tilde{\phi}'_{j,k}(x)\tilde{\phi}'_{j,\ell}(x)dx, \quad k,\ell = 0,1,2,\cdots,2^j-1. \\ \phi_a &= \mathbf{col}\ (\tilde{\phi}_{j,k}(0) : k = 0,1,2,\cdots,2^j-1). \\ \phi_b &= \mathbf{col}\ (\tilde{\phi}_{j,k}(1) : k = 0,1,2,\cdots,2^j-1). \\ \mathbf{c} &= \mathbf{col}\ (c_{j,k} : k = 0,1,2,\cdots,2^j-1). \\ \mathbf{c}_f &= \mathbf{col}\ \Big(\int_c^d f(x)\tilde{\phi}_{j,k}(x)dx : k = 0,1,2,\cdots,2^j-1 \Big). \end{aligned}$$

In this case,

$$\begin{aligned} \lambda_a &= \widehat{w}'(0^-) - \widehat{w}'(0^+) \\ \lambda_b &= \widehat{w}'(1^-) - \widehat{w}'(1^+) \end{aligned}$$

we take as unknowns.

5.1.2 Formulation on domain directly

This approach is much simpler than the embedding domain. We take the space of test functions as :

$$H^1_{per}(0,1) = \{v \in H^1(0,1) : v(0) = v(1)\}.$$

Using the ansatz as previously, we obtain the system of linear equation :

$$\begin{pmatrix} \mathbf{M} & -\phi_0 \\ -\phi_0^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{c}_f \\ \mathbf{0} \end{pmatrix} \tag{5.4}$$

where $\mathbf{M}$, $\mathbf{c}$ dan $\mathbf{c}_f$ coincide to (5.3) and $\phi_0 = \mathbf{col}\ (\tilde{\phi}_{j,k}(0) : k = 0,1,2,\cdots,2^j - 1)$. Here we take $\lambda = u'(1^-) - u'(0^+)$ as an unknown.

Recall that if $\lambda = 0$, namely the solution has a periodic derivatif, then (5.4) is a special case of system introduced by Nielson (1998) as well as Kucera and Vicek (1999).

The structure of the matrices coefficient on the system (5.3) and (5.4) is shown on Figure 5.1.

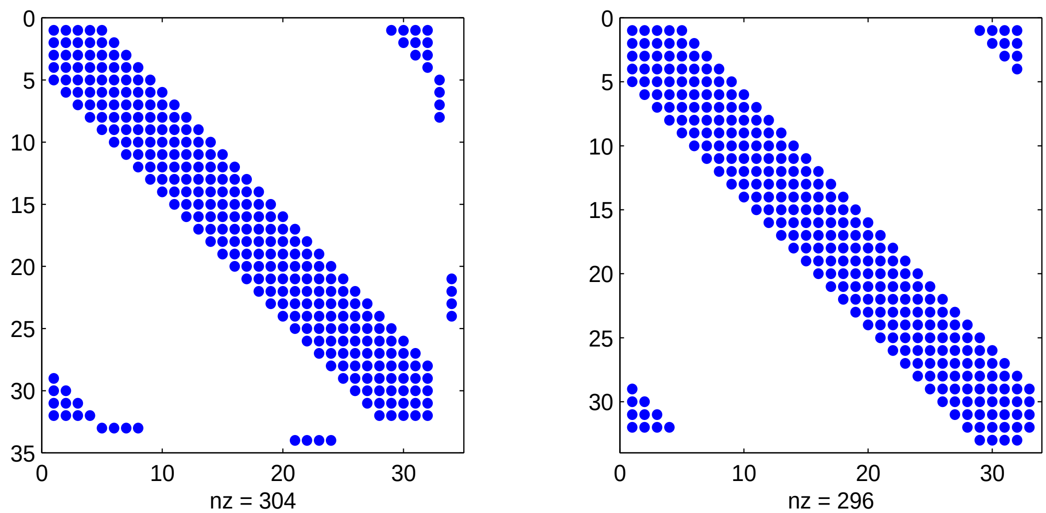

Figure 5.1: The structure of matrices on the system (5.3) (left) and system (5.4) (right).

5.2 The wavelet collocation method for solving BVP

Consider the following BVP :

$$\begin{cases} -u'' + \alpha u = f \text{ on } (0,1) \\ + \textbf{ boundary condition}. \end{cases} \tag{5.5}$$

The boundary condition involves the type of *Dirichlet* : $u(0) = u_0$, $u(1) = u_1$, *Neumann*: $u'(0) = u_0$, $u'(1) = u_1$, and *Robin*: $u(0) = u_0$, $u'(1) = u_1$. The existence of the unique solution can be checked by Lax-Milgram theorem by making the appropriate test function space.

The BVP (5.5) will be approximated by the wavelet collocation method. We need a number of collocation points in $[0, 1]$. In determining the collocation points we should consider the following matters :

(i) The number of collocation points must be the same as the number of wavelet basis.

(ii) Evaluating the differential operator appearing in the equation on the collocation points should be possible and easy to handle. Indeed, if x_ℓ and $\phi_{j,k}$, $\ell, k = 2-D, \cdots, 2^j-1$ are collocation points and basis elements, respectively, then $\phi_{j,k}(x_\ell), \phi'_{j,k}(x_\ell)$ dan $\phi''_{j,k}(x_\ell)$ should be exists and easy to obtain.

(iii) The coefficient matrices should be sparse, banded and with small condition number.

Since that we choose the collocation points of the form $x_\ell = \ell/2^j$, i.e. the dyadic points on jth level. This choice gives the benefits as :

(i) Evaluating $\phi_{j,k}(x_\ell), \phi'_{j,k}(x_\ell)$ and $\phi''_{j,k}(x_\ell)$ can be done easily and exactly. Even, it is enough to know the values on the integers, since :

$$\phi_{j,k}(x_\ell) = 2^{j/2}\phi(2^j x_\ell - k) = 2^{j/2}\phi(\ell - k)$$

(ii) For each collocation point $x_\ell = \ell/2^j$ there are only $D-2$ basis elements $\phi_{j,k}$ contribute, i.e. $k = \ell + 2 - D, \cdots, \ell - 1$. Figure 6.1 shows the collocation points and contribution of basis elements ($D = 4$). In the illustration, it is clear that only two functions $\phi_{j,k}$ contribute at $x_\ell = \ell/2^j$, namely $k = \ell - 1$ dan $k = \ell - 2$. Since that, the resulted matrices will be banded with bandwidth $D - 2$.

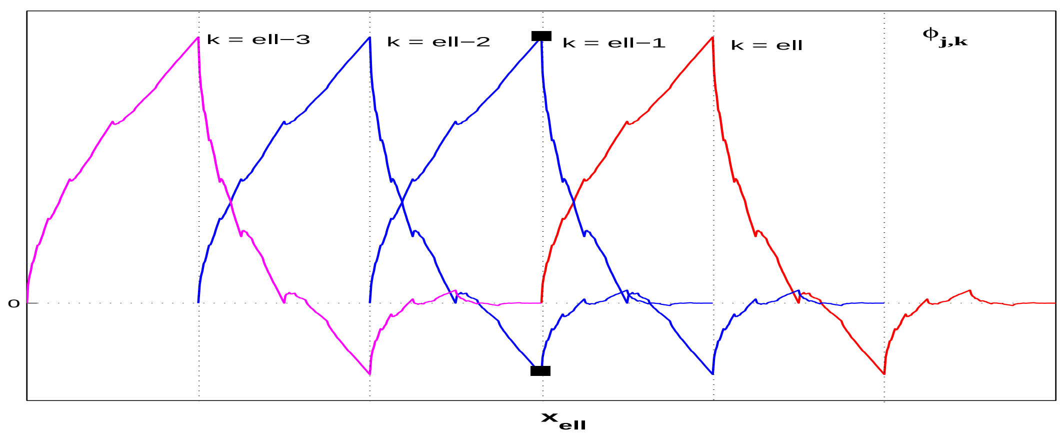

Figure 5.2: Example of collocation points and contribution of basis.

Unfortunately, there are only 2^j dyadic points $x_\ell = \ell/2^j$ on $[0, 1]$ meanwhile the number of restricted wavelet basis is $2^j + D - 2$. In general, at the same level, the number of basis elements is more than the number of dyadic points. Thus, we need the additional collocation points.

Discretization :

Let $\bar{\phi}_{j,k}$, $k = 2 - D, \cdots, 2^j - 1$ be the restricted wavelet basis on the level j. We take an ansatz of the form :

$$u_j(x) = \sum_{k=2-D}^{2^j-1} c_{j,k}\bar{\phi}_{j,k}(x). \tag{5.6}$$

Substituting (5.6) into the differential equation of (5.5) we obtain

$$\sum_{k=2-D}^{2^j-1} c_{j,k}\big[-\bar{\phi}''_{j,k}(x) + \alpha\bar{\phi}_{j,k}(x)\big] = f(x). \tag{5.7}$$

The steps are following :

1. Take the dyadic points $\ell/2^j$, $\ell = 0, 1, \cdots, 2^j$ and insert into (5.7), we obtain

$$\sum_{k=2-D}^{2^j-1} c_{j,k}\big[-\bar{\phi}''_{j,k}(\ell/2^j) + \alpha\bar{\phi}_{j,k}(\ell/2^j)\big] = f(\ell/2^j),\ \ell = 0, 1, \cdots, 2^j.$$

2. In order that boundary condition is fulfilled by ansatz (5.6) the following condition should be satisfied, i.e.

$$\sum_{k=2-D}^{2^j-1} c_{j,k}(\mathbf{B_0}\phi_{j,k})(0) = u_0 \text{ dan } \sum_{k=2-D}^{2^j-1} c_{j,k}(\mathbf{B_1}\phi_{j,k})(1) = u_1$$

   where $\mathbf{B_0}$ and $\mathbf{B_1}$, respectively the operator of boundary condition at $x = 0$ dan $x = 1$.

3. So far we already have the system of $2^j + D - 2$ unknowns with $2^j + 3$ equations. We still need $D - 5$ equations more. For this purpose, we should take take more as $D - 5$ collocation points. We introduce two strategies in determining such collocation points :

   (i) **Strategy 1 :** The additional collocation points are taken as mid-points of each subinterval as $D - 5$, namely

$$\bar{x}_\ell = \ell/2^{j+1}, \ell = 1, 2, \cdots, D - 5.$$

   (ii) **Strategy 2 :** All of the additional collocation points only from the first subinterval, namely

$$\bar{x}_\ell = \ell/2^{j+q}, \ell = 1, 2, \cdots, D - 5$$

   where q some integer such that the subinterval $[0, \frac{1}{2^j}]$ contains at least $D - 5$ dyadic points on the level $j + q$. In this case $q = [|^2 \log(D - 4)|]$, with $[|x|]$ stands for the greatest integer less than or equal to x.

4. Rearranging the system in the natural order we obtain

$$\mathbf{Ac_j} = \mathbf{F} \tag{5.8}$$

   where the first and the last row of $\mathbf{A}$ corresponds to the boundary condition at 0 and 1. The others are :

$$(\mathbf{A})_{\ell,k} = -\phi''_{j,k}(z_\ell) + \alpha\phi_{j,\ell}(z_\ell)$$

   where z_ℓ consists of the ordered x_ℓ and $\bar{x}_\ell$,

$$\mathbf{F} = \begin{pmatrix} u_0 \\ \mathbf{F^o} \\ u_1 \end{pmatrix}$$

   where $\mathbf{F^o} =$ col$(f(z_\ell))$, and $\mathbf{c_j}$ is a vector containing the coeffisients $c_{j,k}$.

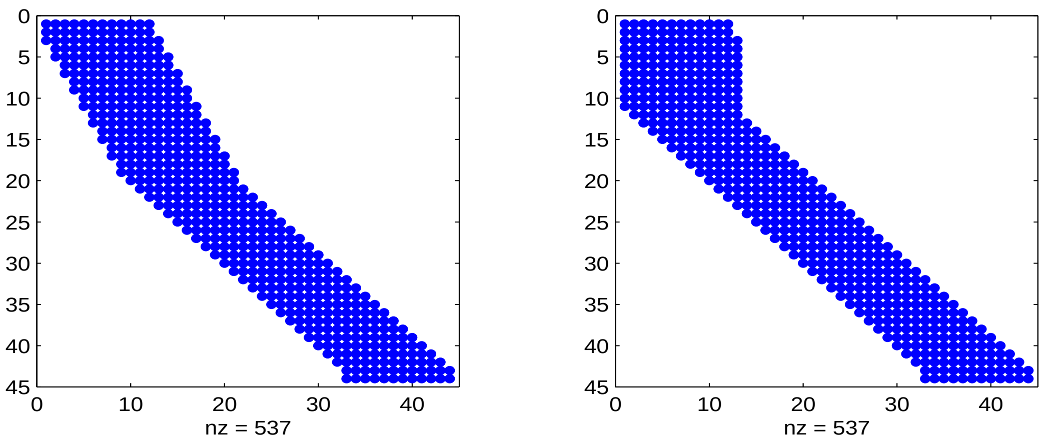

Figure 5.3: Structure of matrices on system (5.8): strategy 1 (left) and strategy 2 (rigtht).

The boundary operator $\mathbf{B}_0$ dan $\mathbf{B}_1$ involve the Dirichlet type : $\mathbf{B}_0 u = \mathbf{B}_1 u = u$, Neumann type : $\mathbf{B}_0 u = \mathbf{B}_1 u = u'$ dan Robin type: $\mathbf{B}_0 u = u$, $\mathbf{B}_1 u = u'$. Because of its simplicity, the scheme can be applied easily to more general the linear differential equations of second order, i.e.

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \text{ on } (0,1) \\ + \textbf{boundary condition}. \end{cases}$$

5.3 The semi-discrete wavelet collocation method for solving IBVP

5.3.1 The parabolic equation

Consider the IVBP of parabolic type as follows

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = f(x,t), & (x,t) \in \Omega \times [0,T] \\ u(x,0) = u_0(x), & x \in \Omega \\ u(0,t) = a, \;\; u(1,t) = b, & t \in [0,T] \end{cases} \tag{5.9}$$

where $\Omega = (0,1)$. The solution of (5.9) is approximated by the ansatz defined by

$$u_j(x,t) = \sum_{k \in \Lambda_j} c_{j,k}(t)\phi_{j,k}(x) \tag{5.10}$$

Here, the coefficients $c_{j,k}$ are functions of time variable t. The approximation scheme is based on the discretization order between time and spatial variable, and usually be called semi discrete approximation. In addition, the solution of (5.9) does not need to be separable.

Discretization of spatial variable

Assume : $\phi_{j,k}$ are periodic wavelet basis $\tilde{\phi}_{j,k}$ and the boundary conditions should be periodic, i.e.

$$u(0,t) = u(1,t) \text{ dan } u'(0,t) = u'(1,t).$$

Steps :

1. Substitute (5.10) into (5.9), we obtain

$$\sum_{k=0}^{2^j-1} c'_{j,k}(t)\tilde{\phi}_{j,k}(x) = \sum_{k=0}^{2^j-1} c_{j,k}(t)\left[\tilde{\phi}''_{j,k}(x) - \tilde{\phi}_{j,k}(x)\right] + f(x,t) \quad (5.11)$$

$$\sum_{k=0}^{2^j-1} c_{j,k}(0)\tilde{\phi}_{j,k}(x) = u_0(x). \quad (5.12)$$

   Since the periodic wavelet basis is orthonormal, we have

$$c_{j,k}(0) = (u_0, \tilde{\phi}_{j,k}) = \int_0^1 u_0(x)\tilde{\phi}_{j,k}(x)dx. \quad (5.13)$$

2. Take the collocation points $x_\ell = \ell/2^j$, $\ell = 0,1,2,\cdots,2^j-1$, then insert into (5.12) and (5.13) so that we find the following initial value problem :

$$\begin{cases} \mathbf{A}\mathbf{c}'(t) = \mathbf{B}\mathbf{c}(t) + \mathbf{F}(t) \\ \mathbf{c}(0) = \mathbf{c}_0 \end{cases} \quad (5.14)$$

   where

$$(\mathbf{A})_{\ell,k} = \tilde{\phi}_{j,k}(x_\ell)\ ,\ (\mathbf{B})_{\ell,k} = \tilde{\phi}''_{j,k}(x_\ell) - \tilde{\phi}_{j,k}(x_\ell), \ell, k = 0,1,2,\cdots,2^j-1$$
$$\mathbf{c}(t) = \mathbf{col}\ [c_{j,k}(t) : k = 0,1,2,\cdots,2^j-1]$$
$$\mathbf{F}(t) = \mathbf{col}\ [f(x_\ell,t) : \ell = 0,1,2,\cdots,2^j-1].$$

   The initial condition $\mathbf{c}_0$ is given by (5.13). It can be written as

$$\begin{cases} \mathbf{c}'(t) = \mathbf{B_1}c(t) + \mathbf{F_1}(t) \\ \mathbf{c}(0) = \mathbf{c}_0 \end{cases} \quad (5.15)$$

   where $\mathbf{B_1} = \mathbf{A^{-1}B}$ dan $\mathbf{F_1} = \mathbf{A^{-1}F}$.

The problem (5.15) may be solved by any standard method, e.g. formula Runge-Kutta.

Discretization of time variabel

Assumption: $\phi_{j,k}$ are restricted wavelet basis $\bar{\phi}_{j,k}$ arising the B-spline cubic.

Steps :

1. The time interval $[0, T]$ to be partitioned into p subintervals $I_1, I_2, \cdots I_N$ with mesh $\triangle t$, $I_k = [t_{k-1}, t_k]$.

2. If $z_k(x) = u(x, t_k)$ then

$$\frac{z_k(x) - z_{k-1}(x)}{\triangle t} = \frac{u(x,t_k) - u(x,t_{k-1})}{\triangle t} \rightarrow \frac{\partial u(x,t)}{\partial t}|_{t=t_k} \text{ if } \triangle t \rightarrow 0.$$

For abbreviation, write $A = -\frac{\partial^2}{\partial x^2} + I$. Thus, for each $k = 1, 2, \cdots, N$, the equation of (5.9) becomes

$$Az_k + \frac{1}{\triangle t} z_k = \frac{1}{\triangle t} z_{k-1} + f_k \tag{5.16}$$

where $f_k(x) = f(x, t_k)$.

3. Together with boundary condition

$$z_k(0) = a \text{ dan } z_k(0) = b \tag{5.17}$$

this equation should be solved iteratively for $z_1, z_2, \cdots, z_N$ with initial value is given by

$$z_0(x) = u(x, 0) = u_0(x).$$

4. Substitute the ansatz (5.10) at $t = t_n$ into (5.16) we obtain

$$\sum_{k\in\Lambda_j} c_{j,k}(t_n)\big[(1 + \Delta t)\phi_{j,k}(x) - (\Delta t)\phi''_{j,k}(x)\big]$$
$$= (\Delta t) f(x, t_n) + \sum_{k\in\Lambda_j} c_{j,k}(t_{n-1})\phi_{j,k}(x) \tag{5.18}$$

and (5.17) becomes

$$\sum_{k\in\Lambda_j} c_{j,k}(t_n)\phi_{j,k}(0) = a \text{ dan } \sum_{k\in\Lambda_j} c_{j,k}(t_n)\phi_{j,k}(1) = b. \tag{5.19}$$

5. Take $x_\ell = \frac{\ell}{2^j}$, $\ell = 0, 1, \cdots, 2^j$ as collocation points, then insert to (5.18) and (5.19) so that we obtain

$$\mathbf{PC_j}(t_n) = \mathbf{F_n} + \mathbf{QC_j}(t_{n-1}) \tag{5.20}$$

where

$$\mathbf{C_j}(t_n) = \mathbf{col}\ [c_{j,k}(t_n) : k = -2, -1, 0, 1, \cdots, 2^j - 1].$$

For $k = -2, -1, 0, 1, \cdots, 2^j - 1,\ \ell = 1, 2, 3, \cdots, 2^j + 1,$ the matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{F}_n$ are defined as follows :

$$\begin{aligned}
(\mathbf{P})_{\ell,k+3} &= (1+\Delta t)\phi_{j,k}(x_{\ell-1}) - (\Delta t)\phi''_{j,k}(x_{\ell-1}),\\
(\mathbf{F}_n)_\ell &= (\Delta t) f(x_{\ell-1}, t_n),\\
(\mathbf{Q})_{\ell,k+3} &= \phi_{j,k}(x_{\ell-1}).\\
(\mathbf{P})_{1,k} &= \phi_{j,k}(0),\ (\mathbf{P})_{2^j+3,k} = \phi_{j,k}(1)\\
(\mathbf{F}_n)_1 &= b_0,\ (\mathbf{F}_n)_{2^j+3} = b_1\\
(\mathbf{Q})_{1,k} &= (\mathbf{Q})_{2^j+3,k} = 0.
\end{aligned}$$

For the other boundary conditions, we needs to modify only the first and the last rows of $\mathbf{P}$.

Hence, the equation (5.20) need to be solved recursively with initial $\mathbf{C_j}(0)$ determined by

$$\sum_{k\in\Lambda_j} c_{j,k}(0)\phi_{j,k}(x) = u_0(x). \tag{5.21}$$

The equation (5.21) could be solved by collocation approach by taking two extra collocation points, e.q. $x_a = 1/2^{j+1}$, $x_b = 1 - 1/2^{j+1}$. Using the the standard procedure we get

$$\mathbf{A}\mathbf{C_j}(0) = \mathbf{U}. \tag{5.22}$$

5.3.2 The Burger equation

The Burger equation is a nonlinear partial differential containing the initial condition, i.e.

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} &= 0,\ x \in (0,1), t > 0 && (5.23)\\
u(x,0) &= u_0(x),\ x \in [0,1] && (5.24)
\end{aligned}$$

where ν a positive parameter. The analytic solution of (5.23) with initial value (5.24) is given by (Logan, 1994) :

$$u(x,t) = \frac{\int_{-\infty}^{\infty}(\frac{x-\xi}{t})e^{-G(\xi,x,t)/2\nu}d\xi}{\int_{-\infty}^{\infty}e^{-G(\xi,x,t)/2\nu}d\xi} \tag{5.25}$$

where

$$G(\xi,x,t) = \frac{(x-\xi)^2}{2t} + \int_0^\xi u_0(s)ds.$$

Now, consider a non homogen Burger equation subject to boundary conditions as follows :

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} &= f(x,t),\ x \in (0,1), t > 0, && (5.26)\\
u(x,0) &= u_0(x),\ x \in [0,1], && (5.27)\\
u(0,t) = a, u(1,t) &= b,\ t > 0. && (5.28)
\end{aligned}$$

For each $n = 1, 2, \cdots, N$, substitute ansatz (5.10) at $t = t_n$ into (5.26), we obtain

$$\sum_{k\in\Lambda_j} c_{j,k}(t_n)\big[\phi_{j,k}(x) - (\Delta t)\nu\phi''_{j,k}(x)\big] +$$
$$(\Delta t)\sum_{k\in\Lambda_j} c_{j,k}(t_n)\phi_{j,k}(x)\sum_{k\in\Lambda_j} c_{j,k}(t_n)\phi'_{j,k}(x) =$$
$$(\Delta t)f(x,t_n) + \sum_{k\in\Lambda_j} c_{j,k}(t_{n-1})\phi_{j,k}(x) \tag{5.29}$$

where $\Lambda_j = \{-2, -1, 0, 1, \cdots, 2^j - 1\}$. Following the same procedure when we built (5.20), we obtain a nonlinear system of equation as follows :

$$\mathbf{PC_j}(t_n) + (\Delta t)\big[\mathbf{QC_j}(t_n)\big]\big[\mathbf{RC_j}(t_n)\big] = (\Delta t)\mathbf{F_n} + \mathbf{QC_j}(t_{n-1}). \tag{5.30}$$

$\mathbf{P}$ is defined by the same rule as $\mathbf{P}$ in (5.20), but the entries are determined by $(\phi_{j,k} - (\Delta t)\nu\phi''_{j,k})$, $\mathbf{Q}$ and $\mathbf{F_n}$ are coincide with definition in (5.20), $\mathbf{R}$ is defined as $\mathbf{Q}$ but the entries are obtained by $\phi'_{j,k}$.

If $\mathbf{C_j}(t_n) = \mathbf{C_n}$ then (5.30) should be done iteratively for $\mathbf{C_1}, \mathbf{C_2}, \cdots, \mathbf{C_N}$ with initial value $\mathbf{C_0}$ obtained by the relation

$$\sum_{k\in\Lambda_j} c_{j,k}(0)\phi_{j,k}(x) = u_0(x).$$

For each $n = 1, 2, \cdots, N$, to solve the equation (5.30) is equivalent to look for the roots of :

$$\mathbf{G_n}(\mathbf{C_n}) = 0$$

where $\mathbf{G_n} : \mathbb{R}^{\#(\Lambda_j)} \longrightarrow \mathbb{R}^{\#(\Lambda_j)}$ defined as

$$\mathbf{G_n}(\mathbf{C_n}) = \mathbf{PC_n} + (\Delta t)[\mathbf{QC_n}][\mathbf{RC_n}] - (\Delta t)\mathbf{F_n} - \mathbf{QC_{n-1}} \tag{5.31}$$

where $\mathbf{C_n} = \mathbf{col}\ [c_{j,k}(t_n) : k \in \Lambda_j]$.

5.4 The wavelet collocation method for solving Fredhlom integral equation

The Fredhlom integral equation of second kind has the form :

$$\lambda u(x) - \int_0^1 k(x,\xi)u(\xi)d\xi = f(x)\ ,\ x \in [0,1] \tag{5.32}$$

where $\lambda \neq 0$ and $k(x,\xi)$ some given function which is called the kernel. For simplicity, (5.32) is written as

$$\lambda u - Ku = f \text{ on } [0,1]$$

where K the integral operator defined as

$$Ku(x) := \int_0^1 k(x,\xi)u(\xi)d\xi. \tag{5.33}$$

5.4.1 Discretization using periodized wavelets basis

Assume that the kernel $k(x,\xi)$ is periodic with respect to variable ξ, i.e.

$$k(x,0)=k(x,1) \text{ for each } x\in[0,1].$$

Take the ansatz of the form

$$u(x)=\sum_{k=0}^{2^j-1} c_{j,k}\tilde{\phi}_{j,k}(x) \tag{5.34}$$

where $\tilde{\phi}_{j,k}$ are the periodized wavelet.

Steps :

1. Substitute the ansatz into (5.32) we obtain

$$\sum_{k=0}^{2^j-1} c_{j,k}\left[\lambda\tilde{\phi}_{j,k}(x)-\int_0^1 k(x,\xi)\tilde{\phi}_{j,k}(\xi)d\xi\right]=f(x).$$

2. Take the collocation points $x_\ell=\ell/2^j$, $\ell=0,1,\cdots,2^j-1$. For each the collocation point x_ℓ, the kernel $k(x_\ell,\xi)$ is represented in the wavelet basis as

$$k(x_\ell,\xi)=\sum_{m=0}^{2^j-1} d^\ell_{j,m}\tilde{\phi}_{j,m}(\xi)$$

where the coefficients $d^\ell_{j,m}$ are given by

$$d^\ell_{j,m}=\int_0^1 k(x_\ell,\xi)\tilde{\phi}_{j,m}(\xi)d\xi.$$

Substituting the collocation points and using the orthonormality of wavelets basis we get

$$d^\ell_{j,m}=\int_0^1 k(x_\ell,\xi)\widetilde{\phi}_{j,m}(\xi)d\xi.$$

3. For each $\ell=0,1,2,\cdots,2^j-1$, the collocation point x_ℓ to be inserted into equation in step 1. Using the orthonormal basis of periodic wavelets we get the relation

$$f(x_\ell) \;=\; \sum_{k=0}^{2^j-1} c_{j,k}\left[\lambda\phi_{j,k}(x_\ell)-d^\ell_{j,k}\right].$$

Thus, we have a linear system of equation as follows :

$$\mathbf{MC_j}=\mathbf{F} \tag{5.35}$$

where

$$\begin{aligned}\mathbf{C_j} &= \mathbf{col}\;[c_{j,k}:k=0,1,\cdots,2^j-1]\\ \mathbf{F} &= \mathbf{col}\;[f(x_\ell):\ell=0,1,\cdots,2^j-1]\\ (\mathbf{M})_{\ell,k} &= (\lambda\tilde{\phi}_{j,k}(x_\ell)-d^\ell_{j,k}),\ell,k=0,1,\cdots,2^j-1.\end{aligned}$$

5.4.2 Discretization using the periodized wavelets basis

Let $\phi_{j,k}$, $k = 2 - D, \cdots, 2^j - 1$ be the restricted wavelets on the level j. Define the ansatz as

$$u_j(x) = \sum_{k=2-D}^{2^j-1} c_{j,k}\bar{\phi}_{j,k}(x).$$

We choose as $2^j + D - 2$ the collocation points consisting the main collocation points and the additional collocation points (if any). Substituting the ansatz into (5.32), then inserting those collocation points we obtain

$$f(x_\ell) = \sum_{k=2-D}^{2^j-1} c_{j,k}\left[\lambda\bar{\phi}_{j,k}(x_\ell) - \int_0^1 k(x_\ell,\xi)\bar{\phi}_{j,k}(\xi)d\xi\right].$$

The value

$$\int_0^1 k(x_\ell,\xi)\bar{\phi}_{j,k}(\xi)d\xi.$$

should be calculated using some quadrature formulae, e.g. trapezium rule. Parallel to (5.35), we obtain the following system :

$$\hat{\mathbf{M}}\hat{\mathbf{C}}_{\mathbf{j}} = \hat{\mathbf{F}}. \tag{5.36}$$

where

$$\begin{aligned}
\hat{\mathbf{C}}_{\mathbf{j}} &= \mathbf{col}\ [c_{j,k} : k = 2 - D, \cdots, 2^j - 1] \\
\hat{\mathbf{F}} &= \mathbf{col}\ [f(x_\ell) : \ell = 2 - D, \cdots, 2^j - 1] \\
(\hat{\mathbf{M}})_{\ell,k} &= \lambda\bar{\phi}_{j,k}(x_\ell) - d^\ell_{j,k}, \ell, k = 2 - D, \cdots, 2^j - 1.
\end{aligned}$$

6 Conclusions and Open Problems

We successfully construct several approximation schemes for solving the operator equations using the wavelet projection method, together with theoretically justification and numerically realization.

Some cases we carry over on the numerical experiments including the BVP of various type of BC, i.e. periodic, Dirichlet, Neumann, Robin. We also take the BVP in distribution, the IBVP of parabolic type and the Burger equation. Furthermore, we adjust the method for solving the Fredhlom integral equation arising from BVP, the case where the kernel periodic as well as non periodic, the equation which has a discontinuous solution.

Based on the results we obtain that the wavelet projection method gives some benefits in accurately, convergence rate and locality the errors. The deficiency of the method is the structure of the matrices are not as good as the classical method such as the finite element and finite difference method. In addition, the wavelet basis in general is not interpolation like the finite element basis.

Some open problems we find out in this research are following :

1. Is it possible to create the wavelet basis on the interval which has the properties : orthogonal, narrow support, higher vanishing moment, and interpolation !.
2. How to develop the scheme such that the fast wavelet transform is applicable!
3. How do we apply the better supporting computation method to do numerical realization in our method.
4. Is it possible that the problem is solved locally, then we combine the solution to obtain the global solution. The finite element method and the multigrid method may be some inspirations.
5. How do we develop the schemes for problems in dimension 2 or 3.

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